Example Sheet 2

Questions marked with * are optional. Please hand in Questions 5 and 7.

1. Show that for a lazy, irreducible, reversible Markov chain, we have that $\gamma = \gamma_*$ and so $t_{\text{rel}} = \frac{1}{\gamma}$. Show also that for a lazy chain with laziness parameter α on state space S of size $n > 1$ (i.e. the chain with transition matrix $\alpha I + (1-\alpha)P$ where P is the transition matrix of some Markov chain), we have that $\frac{1}{\gamma} \geq \frac{n-1}{(1-\alpha)n}$ and $t_{\text{rel}} \geq \frac{n-1}{(1-\alpha)n}$.

2. Suppose that P is irreducible and for every state x consider the set $T(x) = \{t : P^t(x,x) > 0\}$. Show that $T(x) \subseteq 2\mathbb{Z}$ if and only if -1 is an eigenvalue of P.

3. Let P be an irreducible, reversible, and transitive transition matrix on a finite state space S with invariant distribution π. Let λ be an eigenvalue of P with multiplicity one and with associated eigenfunction f which is scaled such that $\max_{x \in S} |f(x)| = 1$.

   (a) Show that $f(x) \in \{-1, 1\}$ for all $x \in S$.

   (b) Show that if P is the transitive transition matrix of a simple random walk on a graph whose vertices have degree d, then there exists $p \in \{0, 1, \ldots, d\}$ such that
   $$\lambda = 1 - \frac{2p}{d}.$$

   (c) For each of the following items, find a (possibly lazy) random walk on a graph $G = (V, E)$ with $|V| \geq 3$ and whose vertices have degree $d \geq 2$ verifying:

      i. There exists $p \in \{1, \ldots, d-1\}$ such that $1 - \frac{2p}{d}$ is an eigenvalue with multiplicity one.

      ii. 1 and -1 are the unique eigenvalues with multiplicity one.

      iii. 1 is the unique eigenvalue with multiplicity one.

4. * Consider a Markov chain on $S = \{1, \ldots, n\}$ with transition matrix P defined as follows.
   $$P(i,j) = \begin{cases} \frac{1}{i(i+1)}, & \text{for } i \leq n-1 \text{ and } j \leq i, \\ \frac{i}{i+1}, & \text{for } i \leq n-1 \text{ and } j = i+1, \\ \frac{1}{n}, & \text{for } i = n \text{ and } j \leq n. \end{cases}$$
   Define the relaxation time to be $t_{\text{rel}} = (1 - \max\{|\lambda| : \lambda \neq 1 \text{ is an eigenvalue of } P\})^{-1}$ (notice that this definition agrees with the one for reversible chain). Find the order of the relaxation time of P, its invariant distribution and show that $t_{\text{mix}} \gtrsim n$. Hence conclude that the upper bound on the mixing time in terms of the relaxation time from Theorem 4.10 does not hold for this chain for large enough n, showing that the reversibility assumption in the theorem (and therefore also in the Poincaré inequality) is necessary.

5. (a) Let X be a reversible Markov chain on S with transition matrix P and invariant distribution π. Prove a generalisation of the Poincaré inequality, i.e., for all $f : S \to \mathbb{R}$, show that
   $$\text{Var}_\pi\left(P^t f\right) \leq e^{-2t/t_{\text{rel}}} \text{Var}_\pi(f).$$

(b) Show that without the reversibility assumption, if $\gamma(Q)$ denotes a Poincaré constant of Q, for all $f : S \to \mathbb{R}$, we have

$$\mathrm{Var}_\pi\left(Pf\right) \leq \left[1 - \gamma(P^*P)\right] \mathrm{Var}_\pi\left(f\right).$$

Moreover, if P is lazy show that $\gamma(P^*P) \geq \gamma(P)$.

6. * Let P be a reversible transition matrix on a finite state space with invariant distribution π. Define the total variation distance from stationarity from a typical point at time t to be

$$d_{\text{ave}}(t) = \sum_{x \in S} \pi(x) \|P^t(x, \cdot) - \pi\|_{\text{TV}}.$$

Suppose that $1 = \lambda_1 \geq \ldots \geq \lambda_n \geq -1$ are the eigenvalues, then show that

$$4 d_{\text{ave}}(t)^2 \leq \sum_{j=2}^{n} \lambda_j^{2t}.$$

7. Consider a lazy simple random walk on a cycle $\mathbb{Z}_n$. Show that for all $\alpha > 0$ we have as $n \to \infty$

$$d(\alpha n^2) \to \int_0^1 \left| \sum_{k=1}^{\infty} e^{-\alpha \pi^2 k^2} \cos(2\pi k u) \right| du.$$

(Hint: First write

$$d(\alpha n^2) = \frac{1}{2} \int_0^1 \left| 1 - n P^{\lceil \alpha n^2 \rceil}(0, \lfloor un \rfloor) \right| du$$

and then use the spectral theorem.)

8. Let X be a reversible Markov chain on S with transition matrix P and invariant distribution π.

(a) Let ϕ be an eigenfunction of P corresponding to eigenvalue $\lambda \neq 1$ and $\|\phi\|_2 = 1$. Show that

$$\mathbb{E}_\pi\left[\left(\sum_{s=0}^{t-1} \phi(X_s)\right)^2\right] \leq \frac{2t}{1-\lambda}.$$

(b) Let $f : S \to \mathbb{R}$ be a function with $\mathbb{E}_\pi[f] = 0$. Recall $\gamma = 1 - \lambda_2$ is the spectral gap. Show that

$$\mathbb{E}_\pi\left[\left(\sum_{s=0}^{t-1} f(X_s)\right)^2\right] \leq \frac{2t\mathbb{E}_\pi\left[f^2\right]}{\gamma}.$$

(c) Using coupling or otherwise, show that if $r \geq t_{\text{mix}}(\varepsilon/2)$ and $t \geq \frac{4 t_{\text{rel}} \mathrm{Var}_\pi(f)}{\eta^2 \varepsilon}$, then for all $x \in S$,

$$\mathbb{P}_x\left(\left| \frac{1}{t} \sum_{s=0}^{t-1} f(X_{r+s}) - \mathbb{E}_\pi[f] \right| \geq \eta\right) \leq \varepsilon.$$

9. Let X be an irreducible, lazy, and reversible Markov chain on S with transition matrix P and stationary distribution π. Let $1 = \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ be the eigenvalues of P.

(a) Show that

$$\mathbb{E}_\pi[T_\pi] := \sum_{x,y} \pi(x)\pi(y)\mathbb{E}_x[T_y] = \sum_{i \geq 2} \frac{1}{1 - \lambda_i}.$$

(Hint: Use question 12b from the first example sheet.)

(b) Show that
$$\sum_{t=k}^{\infty}(P^t(x,x)-\pi(x)) \le e^{-k/t_{\mathrm{rel}}}\mathbb{E}_\pi[T_x]\,.$$

10. Let X be a reversible Markov chain on a finite state space S with transition matrix P and invariant distribution π. Let $A \subset S$ and let $B = A^c$ with $k = |B|$. Suppose that the sub-stochastic matrix P_B (the restriction of P to B, i.e., $P_B(x,y) = P(x,y)$ for $x,y \in B$) is irreducible, in the sense that for all $x,y \in B$, there exists $t \ge 0$ such that $P_B^t(x,y) > 0$.

   (a) By defining an appropriate inner product, show that P_B has k real eigenvalues $1 > \gamma_1 \ge \gamma_2 \ge \dots \ge \gamma_k$.

   (b) Show that there exist nonnegative numbers $a_1,\dots,a_k$ satisfying $\sum_i a_i = 1$ such that for all $t \ge 0$ we have
   $$P_{\pi_B}(T_A > t) = \sum_{i=1}^{k} a_i\gamma_i^t,$$
   where $\pi_B(x) = \frac{\pi(x)}{\pi(B)}$ for all $x \in B$.

   (c) The Perron-Frobenius theorem gives that $\gamma_1 > 0$ and $\gamma_1 \ge -\gamma_k$. Establish that
   $$\gamma_1 \le 1 - \frac{\pi(A)}{t_{\mathrm{rel}}}$$

   (d) Deduce that
   $$\mathbb{P}_{\pi_B}(T_A > t) \le \left(1 - \frac{\pi(A)}{t_{\mathrm{rel}}}\right)^t \le \exp\left(-\frac{t\pi(A)}{t_{\mathrm{rel}}}\right).$$

   (e) By the Perron-Frobenius theorem, the left eigenvector v corresponding to $\gamma_1 > 0$ is strictly positive. Let α be a probability distribution given by $\alpha = \frac{v}{\sum_i v(i)}$. Show that when the starting distribution is α, then the law of T_A is geometric with parameter γ_1. Prove that for all t and all y,
   $$\mathbb{P}_\alpha(X_t = y \mid T_A > t) = \alpha(y).$$
   Finally, show that if P_B is in addition aperiodic, then for all $x \notin A$ we have
   $$\mathbb{P}_x(X_t = y \mid T_A > t) \to \alpha(y) \text{ as } t \to \infty.$$
   (The distribution α is called the quasi-stationary distribution.)

11. Consider a lazy reversible transition matrix P on S.

   (a) Show that $s(t) \le \max_{x,y\in S}\mathbb{P}_x(T_y > t)$ where T_y is the hitting time of y and $s(t)$ is the separation distance. Conclude that $t_{\mathrm{mix}} \le 4t_{\mathrm{hit}}$, where $t_{\mathrm{hit}} = \max_{x,y\in S}\mathbb{E}_x[T_y]$.

   (b) Using Example Sheet 1, Question 12b show that for all $t > 0$ we have $\frac{\mathbb{E}_\pi[T_x]}{t} \ge \frac{P^t(x,x)}{\pi(x)} - 1$ and conclude that $t_{\mathrm{mix}}^{(\infty)} \le 4t_{\mathrm{hit}}$, which, as $t_{\mathrm{mix}} \le t_{\mathrm{mix}}^{(2)}(1/2) = \lceil t_{\mathrm{mix}}^{(\infty)}(1/4)/2\rceil$ also shows that $t_{\mathrm{mix}} \le 2t_{\mathrm{hit}} + 1$.

12. Show that for an irreducible Markov chain P on S with stationary distribution π, the target time $t_{\odot}^a = \sum_{x\in S}\mathbb{E}_a[T_x]\,\pi(x)$ does not depend on $a \in S$. Deduce that $t_{\mathrm{hit}} \le 2\max_{x\in S}\mathbb{E}_\pi[T_x]$.

13. Let X be a lazy biased walk on $\{0,\dots,n\}$ with $P(i,i+1) = \frac{1}{3} = \frac{1}{2} - P(i,i-1)$ for $i \in \{1,\dots,n-1\}$ and $P(i,i) = \frac{1}{2}$ and $P(n,n-1) = P(0,1) = \frac{1}{2}$. Show that X exhibits cutoff at $6n$, i.e., that for all $\varepsilon \in (0,1)$ we have as $n \to \infty$
$$t_{\mathrm{mix}}(\varepsilon) = 6n(1+o(1)).$$