# Example Sheet 1

Questions marked with * are optional. Please hand in Questions 9 and 13.

1. Let $\Omega = \prod_{i=1}^n \Omega_i$, where $\Omega_i$ are finite sets. For each $i$, let $\mu_i$ and $\nu_i$ be probability distributions on $\Omega_i$ and set $\mu = \prod_{i=1}^n \mu_i$ and $\nu = \prod_{i=1}^n \nu_i$. Show that
$$\|\mu - \nu\|_{\mathrm{TV}} \leq \sum_{i=1}^n \|\mu_i - \nu_i\|_{\mathrm{TV}}.$$

2. Let $X$ and $Y$ be Poisson random variables with parameters $\lambda$ and $\mu$ respectively. Writing $\mathcal{L}(X)$ and $\mathcal{L}(Y)$ for their laws, prove that
$$\|\mathcal{L}(X) - \mathcal{L}(Y)\|_{\mathrm{TV}} \leq |\lambda - \mu|.$$

3. Let $Y$ be a random variable with values in $\mathbb{N}$ which satisfies $\mathbb{P}(Y = j) \leq c$ for all $j > 0$ and $\mathbb{P}(Y = j)$ is decreasing in $j$, where $c$ is a positive constant. Let $Z$ be an independent random variable with values in $\mathbb{N}$. Prove that
$$\|\mathbb{P}(Y + Z = \cdot) - \mathbb{P}(Y = \cdot)\|_{\mathrm{TV}} \leq c\mathbb{E}[Z]\,.$$

4. Let $X$ be a Markov chain and let $W$ and $V$ be random variables taking values in $\mathbb{N}$ and suppose they are independent of $X$. Prove that
$$\|\mathbb{P}(X_W = \cdot) - \mathbb{P}(X_V = \cdot)\|_{\mathrm{TV}} \leq \|\mathbb{P}(W = \cdot) - \mathbb{P}(V = \cdot)\|_{\mathrm{TV}}.$$

5. * Let $\mu$ and $\nu$ be two probability distributions on $S$, and let $f$ be a real-valued function on $S$. If
$$|\mathbb{E}_\mu[f] - \mathbb{E}_\nu[f]| \geq r\sigma,$$
where $\sigma^2 = \frac{1}{2}[\mathrm{Var}_\mu(f) + \mathrm{Var}_\nu(f)]$, then show that
$$\|\mu - \nu\|_{\mathrm{TV}} \geq 1 - \frac{4}{4 + r^2}.$$

6. Let $X$ be a simple random walk on a graph $G$ and let $Y$ be a lazy simple random walk on $G$. Let $\mathcal{R}_X(t)$ and $\mathcal{R}_Y(t)$ denote the random variables corresponding to the number of different vertices visited by time $t$ by the walks $X$ and $Y$, respectively ($\mathcal{R}$ stands for the range). Show that for any starting distribution $\mu$, and for all $t, m \in \mathbb{N}$ we have $\mathbb{P}_\mu(\mathcal{R}_X(t) > m) \geq \mathbb{P}_\mu(\mathcal{R}_Y(t) > m)$.

7. Let $G = (V, E)$ be a finite connected graph with the maximal distance between any two vertices equal to $D$. Suppose that $X$ is a (possibly lazy) simple random walk on $G$. Prove that for all $\varepsilon < 1/2$ we have
$$t_{\mathrm{mix}}(\varepsilon) \geq D/2.$$

8. Let $X$ be a Markov chain on state space $S$ with transition matrix $P$ and invariant distribution $\pi$. Let $A \subseteq S$ be a subset with $\pi(A) \geq 1/8$. Let $T_A = \inf\{t \geq 0 : X_t \in A\}$. Prove that there exists a positive constant $c$ so that
$$t_{\mathrm{mix}}(1/4) \geq c \max_x \mathbb{E}_x[T_A]\,.$$

9. Let $X$ be a lazy simple random walk on the $d$-dimensional discrete torus $\mathbb{Z}_n^d$. Show that there exists a positive constant $c$ (depending on the dimension $d$) so that

$$t_{\mathrm{mix}}(1/4) \leq cn^2.$$

10. (a) Let $S_n$ be the symmetric group and let $\sigma \in S_n$ be a uniform random permutation. Let $X$ denote the number of fixed points of $\sigma$, i.e. the number of $1 \leq i \leq n$ such that $\sigma(i) = i$. Show that $\mathbb{E}[X] = 1$ and $\mathrm{Var}(X) = 1$.

    (b) Consider the random transposition shuffle as a method of shuffling a deck of $n$ cards. At each step, the shuffler chooses two cards, $L_t$ and $R_t$, independently and uniformly at random. If $L_t$ and $R_t$ are different, then transpose them. Otherwise, do nothing. Prove that for any $\varepsilon > 0$ and all $n$ sufficiently large we have

$$t_{\mathrm{mix}}(1/4) \geq \left(\frac{1}{2} - \varepsilon\right) n \log n.$$

11. * By considering a suitable coupling of the chain starting from the uniform permutation of cards and a chain starting from identity permutation, show that there is a constant $c$ such that the mixing time of a random-to-top card shuffling satisfies $t_{\mathrm{mix}} \leq n \log n + cn$.

12. (a) Let $\tau$ be a stopping time for a finite and irreducible Markov chain satisfying $\mathbb{E}[\tau] < \infty$ and $P_a(X_\tau = a) = 1$. Show that for all $x$ we have

$$\mathbb{E}_a\left[\sum_{t=0}^{\tau-1} 1(X_t = x)\right] = \pi(x)\mathbb{E}_a[\tau]\,.$$

    (b) Consider a finite, irreducible and aperiodic Markov chain. Prove that for all $x$ we have

$$\pi(x)\mathbb{E}_\pi[T_x] = \sum_{t=0}^{\infty}(P^t(x,x) - \pi(x)).$$

    *(Hint: Count the number of visits to $x$ up until $T_x^m = \inf\{t \geq m : X_t = x\}$ in two different ways: using part (a) and also using the convergence to equilibrium theorem.)*

13. Let $P$ be the transition matrix of a finite reversible chain with invariant distribution $\pi$. Using the Cauchy-Schwarz inequality or otherwise prove that for all $x, y$ and all $t$ we have

$$\frac{P^{2t}(x,y)}{\pi(y)} \leq \sqrt{\frac{P^{2t}(x,x)}{\pi(x)} \cdot \frac{P^{2t}(y,y)}{\pi(y)}} \quad \text{and} \quad P^{2t+2}(x,x) \leq P^{2t}(x,x).$$

    Show that if $P$ is lazy, then we have $P^{t+1}(x,x) \leq P^t(x,x)$ for all $t \geq 0$, and deduce that $P^t(x,x) \geq \pi(x)$.

    *(Hint: For the lazy chain consider a suitable chain on the state space where you add a vertex for each $x, y$ such that $P(x,y) > 0$ and apply the previous part.)*

14. Let $X$ be an irreducible Markov chain on $S$ with transition matrix $P$ and invariant distribution $\pi$.

    (a) Recall that the separation distance is $s(t) = \max_{x,y\in S}\left(1 - \frac{P^t(x,y)}{\pi(y)}\right)$. Show that $s(t)$ is non-increasing as a function of $t$.

(b) Define $t_{\mathrm{sep}}(\varepsilon) = \min\{t \geq 0 : s(t) \leq \varepsilon\}$. Show that for all $\varepsilon \in (0, 1]$ and all $k \in \mathbb{N}$ we have that $t_{\mathrm{sep}}(\varepsilon^k) \leq k t_{\mathrm{sep}}(\varepsilon)$.

15. * Let $X$ be a reversible Markov chain on the finite state space $S$ with transition matrix $P$ and invariant distribution $\pi$.

(a) Prove that for all $x, y \in S$ we have

$$\frac{P^{2t}(x,y)}{\pi(y)} \geq \left(1 - \max_{z,w \in S} \|P^t(z,\cdot) - P^t(w,\cdot)\|_{\mathrm{TV}}\right)^2.$$

Deduce that

$$P^{2t_{\mathrm{mix}}}(x,y) \geq \frac{1}{4}\pi(y),$$

and that there exists a transition matrix $\widetilde{P}$ such that

$$P^{2t_{\mathrm{mix}}}(x,y) = \frac{1}{4}\pi(y) + \frac{3}{4}\widetilde{P}(x,y).$$

(b) Define

$$t_{\mathrm{stop}} = \max_{x \in S} \min\{\mathbb{E}_x[\Lambda_x] : \Lambda_x \text{ is a randomized stopping time s.t. } P_x(X_{\Lambda_x} \in \cdot) = \pi(\cdot)\}.$$

By defining an appropriate stationary time, prove that $t_{\mathrm{stop}} \leq 8 t_{\mathrm{mix}}$.

(c) We say that a randomized stopping time $T$ starting from $x$ has a halting state if there exists $z \in S$ such that $T \leq T_z$, where $T_z = \min\{t \geq 0 : X_t = z\}$. Show that if $T$ has a halting state and $\mathbb{P}_x(X_T \in \cdot) = \pi(\cdot)$, then $T$ is mean optimal, in the sense that

$$\mathbb{E}_x[T] = \min\{\mathbb{E}_x[\Lambda_x] : \Lambda_x \text{ is a randomized stopping time s.t. } \mathbb{P}_x(X_{\Lambda_x} \in \cdot) = \pi(\cdot)\}.$$

*(Hint: Consider* $\nu(y) = \mathbb{E}_x\left[\sum_{t=0}^{T-1} \mathbb{1}_{\{X_t = y\}}\right]$*, and compare these for different stopping times* $T$*. Use the uniqueness of the invariant measure up to a multiplicative constant.)*

16. * Recall that one step of the riffle shuffle of $n$ cards sends permutation $\sigma$ to the permutation obtained in the following way. Take a subset $A$ of $\{1, \ldots, n\}$ such that for all $i \leq n$ we have $\mathbb{P}(i \in A) = \frac{1}{2}$ independently of other elements in $A$ and if $|A| = k$ place cards $\sigma_1, \sigma_2 \ldots \sigma_k$ on locations in the set $A$ while keeping their relative order and place cards $\sigma_{k+1}, \ldots, \sigma_n$ on locations which are not in $A$ while also keeping their relative order.

(a) Show that the time reversal $P^*$ of riffle shuffle can be described as a chain which at every step assigns value 0 or 1 to every card and then takes all cards which were assigned value 1 at that time and places them above all cards which were assigned value 0 at that step while keeping the same relative order of the cards which were assigned the same value at that step.

(b) Show that if we keep track of values assigned to the cards in part (a), the first time when all cards have different vectors of zeros and ones assigned to them is the strong stationary time for the reversal chain and use this to show that the mixing time of reversal is, for all sufficiently large $n$, upper bounded by $2\log_2(Cn)$ for some constant $C > 0$.

(c) Show that if $P$ is a transition matrix of a transitive chain with time reversal $P^*$ then

$$\|P^{*t}(x,\cdot) - \pi\|_{\mathrm{TV}} = \|P^t(x,\cdot) - \pi\|_{\mathrm{TV}}$$

and conclude that the mixing time of riffle shuffle is also upper bounded by $2\log_2(Cn)$ for all sufficiently large $n$.

(d) By considering the number of states the chain can reach by time $t$ show a lower bound on mixing time of order $\log_2 n$.