

Groups

All of the questions from this sheet were taken from Imre Leader's introductory sheets.

1 Part 1

1. Is there a non-cyclic group of order 55? Is there a non-cyclic group of order 65?
2. For which natural numbers n is there a unique group of order n ?
3. Write down a finite non-abelian group G in which the probability that two elements (chosen at random, with all $|G|^2$ choices equally likely) commute is less than $1/2$. And write down an example where the probability is greater than $1/2$.
4. Let G be a group, and let H be a proper subgroup of G . Is it possible that G is isomorphic to H ?
5. Let G be a group, and let H be a subgroup of G and x an element of G . If $x^{-1}Hx \subset H$, does it follow that $x^{-1}Hx = H$?
6. Is it true that for every finite group G there exists a subset of $\mathbb{R}^n$ (for some n) whose symmetry group is isomorphic to G ?
7. Let G and H be groups such that $G \times \mathbb{Z}$ is isomorphic to $H \times \mathbb{Z}$. Must G be isomorphic to H ?
8. Let A be a (non-empty) set with an associative binary operation, such that for every $a \in A$ there is a unique $a^* \in A$ with $aa^*a = a$. Prove that A is a group.

2 Part 2

9. Write down a finite group G having a non-normal subgroup of index 5. Is there an example where G has odd order?
10. Does the expected number of cycles in a random permutation on $\{1, 2, \dots, n\}$ tend to infinity as n tends to infinity?
11. Let w be a word in the symbols a and b (and their inverses) that is not (equivalent to) the trivial word. Must there exist a finite group G , and elements a and b of G , for which w is not equal to the identity?
12. Which finite groups have the property that all non-identity elements are conjugate? Is there an infinite group with this property?

13. Which finite groups G have the property that the only automorphism of G is the identity map? Is there an infinite group with this property?
14. Write down a countably infinite group having exactly two automorphisms. Is there an uncountable group with this property?
15. Give explicit real vector spaces V and W and an explicit function f from V to W such that f is additive ($f(x + y) = f(x) + f(y)$ for all x, y) but not linear.