Analysis 1

All of the questions from this sheet were taken from Imre Leader's introductory sheets.

1 Part 1

1. We say that a function f from $\mathbb{R}$ to $\mathbb{R}$ crosses the axis at a point x if $f(x) = 0$ but for any $\varepsilon > 0$ there exist y and z in $(x - \varepsilon, x + \varepsilon)$ with $f(y) > 0$ and $f(z) < 0$. Can a continuous function cross the axis at uncountably many places?

2. Let $[a_1, b_1], [a_2, b_2], \dots$ be a sequence of closed intervals whose union contains $[0, 1]$. Does it follow that $\sum_{n=1}^{\infty}(b_n - a_n) \geq 1$?

3. Construct a continuous function from $[0, 1]$ to $\mathbb{R}^2$ that is exactly 3 – 1 onto its image.

4. Is there a continuous function from $[0, 1]$ to $\mathbb{R}^2$ that is exactly 2 – 1 onto its image?

5. Is every function from $[0, 1]$ to $\mathbb{R}$ the pointwise limit of a sequence of continuous functions?

6. Let f be an infinitely differentiable function from $\mathbb{R}$ to $\mathbb{R}$, such that for every x there is an n with all the derivatives $f^{(n)}(x), f^{(n+1)}(x), f^{(n+2)}(x), \dots$ being zero. Must f be a polynomial?

7. Let f be a dierentiable function from $\mathbb{Q}$ to $\mathbb{Q}$ such that $f' = f$. Must f be identically zero?

8. If f is a polynomial of one real variable that is bounded below (on $\mathbb{R}$), explain why f attains its minimum value. If f is a polynomial of two real variables that is bounded below (on $\mathbb{R}^2$), must f attain its minimum value?

2 Part 2

9. Is there an enumeration of $\mathbb{Q}$ as $q_1, q_2, q_3, \dots$ such that $\sum_{n=1}^{\infty}(q_n - q_{n+1})^2$ is finite?

10. Write down a dierentiable function f from $[0, 1]$ to $\mathbb{R}$ such that f' is not (Riemann) integrable. If f' is bounded, must it be integrable?

11. Find a continuous surjection from $\mathbb{R}$ to $\mathbb{R}^2$.

12. Is there a continuous bijection from $\mathbb{R}$ to $\mathbb{R}^2$?

13. Construct a function from $\mathbb{R}$ to $\mathbb{R}$ that is infinitely-dierentiable, but is identically 1 on $[1, 1]$ and identically 0 outside $(2, 2)$.

14. A subset of $\mathbb{R}$ is called perfect if it is closed and has no isolated points. Write down a (non-empty) perfect set that does not contain any (non-trivial) interval. Prove that every closed set is the union of a perfect set and a countable set.

15. Can a simple closed curve in the plane have positive area?