Recent advances on quantum systems with random Hamiltonians

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co-authors

I'll consider a quantum N-particle Schrödinger operator, in a Euclidean space or on a cubic lattice, with a random external field, for $N > 1$:

$$\mathbf{H}^{(N)}(\omega)f(\mathbf{x}) = -\frac{1}{2}\mathbf{\Delta} f(\mathbf{x}) + \sum_{1\leq i\leq N} V(x_i;\omega)f(\mathbf{x}) + \sum_{1\leq i_1<i_2\leq N} \Phi(|x_{i_1} - x_{i_2}|)f(\mathbf{x}) \qquad (1)$$

where $\mathbf{x} = (x_1, \dots, x_N) \in \mathbb{R}^{d\times N}$ or $\mathbb{Z}^{d\times N}$,
$x_i = (\mathrm{x}_i^1, \dots, \mathrm{x}_i^d) \in \mathbb{R}^d$ or $\mathbb{Z}^d$,
and $f \in L_2(\mathbb{R}^{d\times N})$ or $\ell_2(\mathbb{Z}^{d\times N})$.

E.g., for $N = 2$ (two particles), with $\mathbf{x} = (x_1, x_2)$:

$$\begin{aligned}\mathbf{H}^{(2)}(\omega)f(x_1, x_2) = -\frac{1}{2}\Big(\Delta_1 + \Delta_2\Big)f(x_1, x_2) \\ + \Phi(|x_{i_1} - x_{i_2}|)f(x_1, x_2) \qquad (2) \\ + [V(x_1;\omega) + V(x_2;\omega)]f(x_1, x_2),\ x_1, x_2 \in \mathbb{R}^d \text{ or } \mathbb{Z}^d.\end{aligned}$$

If we take $N = 1$, the operator simplifies:

$$\mathsf{H}^{(1)}(\omega)f(x) = -\frac{1}{2}\Delta f(x) + V(x;\omega)f(\mathbf{x}), \ \ x \in \mathbb{R}^d \text{ or } \mathbb{Z}^d. \quad (3)$$

In (1)–(3), $\mathbf{\Delta} = \sum\limits_{1\leq i\leq N} \Delta_i$ where Δ_i is the Laplacian in

variable x_i from vector $\mathbf{x} = (x_1, \ldots, x_N)$:

$$\Delta_i f(\mathbf{x}) = \sum_{1\leq j\leq d} \frac{\partial^2 f}{\partial \mathrm{x}_i^j \, \partial \mathrm{x}_i^j}(\mathbf{x}) \ \text{ for } \ \mathbf{x} \in \mathbb{R}^{d\times N},$$

$$\Delta_i f(\mathbf{x}) = \sum_{1\leq j\leq d} \Big[f(\mathbf{x} + \mathbf{e}_i^j) + f(\mathbf{x} - \mathbf{e}_i^j) \\ - 2f(\mathbf{x})\big] \ \text{ for } \ \mathbf{x} \in \mathbb{Z}^{d\times N}.$$

Physically speaking, $-\mathbf{\Delta}/2$ represents the kinetic

energy operator for our particle system.

When I consider a system in a 'box', I take

Dirichlet’s boundary conditions.

The randomness in generated via the external potential

$$x \mapsto V(x;\omega).$$

Here ω means a point in a probability space $(\Omega, \mathbb{P})$.

We assume that random variables $V(x;\omega)$ carry a lot of independence. In the lattice case:

$$V(x;\omega) \quad \textbf{are IID for} \quad x \in \mathbb{Z}^d.$$

In a continuous case, we use a 'bump' function $\varphi > 0$:

$$V(x;\omega) = \sum_{s\in\mathbb{Z}^d} \mathsf{V}(s;\omega)\varphi(x-s), \;\; x \in \mathbb{R}^d$$

where

$$\mathsf{V}(s;\omega) \quad \textbf{are IID for} \quad s \in \mathbb{Z}^d.$$

Assume that the random variables have a common PDF p, where $0 < p < \mathsf{B}$ on an interval $(0, \mathsf{A})$ and $p = 0$ on $\mathbb{R} \setminus (0, \mathsf{A})$, for some $\mathsf{A}, \mathsf{B} > 0$.

Finally, Φ is a two-body interaction potential obeying, for some $a, b > 0$:

$$|\Phi(y)| \leq b, \;\; \Phi(y) = 0 \;\; \text{when} \;\; |y| > a, \; y \in \mathbb{R}^d \text{ or } \mathbb{Z}^d.$$

This potential doesn't allow us to split the system into non-interacting parts.

Consequently, we may have strong probabilistic dependence between sums $\sum\limits_{1 \leq i \leq N} V(x_i; \omega)$ and $\sum\limits_{1 \leq i' \leq N} V(x_{i'}; \omega)$ for distant vectors $\mathbf{x}$ and $\mathbf{x}'$.

E.g., for $N = 2$, on a lattice $\mathbb{Z}^d$, values $V(x; \omega)$ are independent for distinct sites $x \in \mathbb{Z}^d$, but if vectors $\mathbf{x} = (x_1, x_2), \mathbf{x}' = (x'_1, x'_2) \in \mathbb{Z}^{d \times 2}$ have entries

$x_i = x_i' = x$ for some $i = 1, 2$, the sums

$$[V(x_1;\omega) + V(x_2;\omega)] \text{ and } [V(x_1';\omega) + V(x_2';\omega)]$$

will contain the same summand $V(x;\omega)$.

This yields dependence, even for $\mathbf{x}$ and $\mathbf{x}'$ far apart.

Of course, for $N = 1$ we do not have these difficulties.

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The random Hamiltonian $\mathbf{H}^{(N)}(\omega)$ describes a quantum system with (classical) impurities. If we don't know the exact nature of these impurities, why not to assume that they are random?

Such an idea goes back to Mott and Anderson, two Nobel Prize laureates, and it was vigorously put forward in the Cavendish Laboratory, in the 1950s and 1960s.

But Mott–Anderson considered **one** particle, with the Hamiltonian $\mathsf{H}^{(1)}$, not $\mathbf{H}^{(N)}$.

For a single-particle model, some impressive

results have been established earlier. E.g.:

Take a one-particle random Hamiltonian in $\ell_2(\mathbb{Z}^d)$*:*

$$\mathsf{H}^{(1)}(\omega) = -\frac{1}{2}\Delta + gV(\,\cdot\,;\omega).$$

Recall, Δ *is the lattice Laplacian: for* $x \in \mathbb{Z}^d$*,*

$$\Delta f(x) = \sum_{1\leq j\leq d} [f(x+e_j) + f(x-e_j) - 2f(x)].$$

Under the above assumptions on variables

$V(x;\omega)$*, take constant* g *large enough* $|g| \geq g^*$*.*

: *Then,* $\mathbb{P}$*-a.s.,* $\mathsf{H}^{(1)}$ *has a pure point spectrum.*

More precisely, $\exists$ *a non-random* $m^* > 0$ *such that,*

for the random eigenfunctions ψ_j of $\mathsf{H}^{(1)}(\omega)$,

$$\Big|\psi_j(x;\omega)\Big| \leq c_j(\omega)\mathsf{e}^{-m^*|x|}, \;\; x \in \mathbb{Z}^d.$$

Such a phenomenon is known as (exponential) Anderson localisation at large disorder. The value m^* is called an (effective) ‘mass’, while $1/m^*$ a ‘localisation length’, in Hamiltonian $\mathsf{H}^{(1)}(\omega)$.

A comprehensible proof appeared in 1989, via the the so-called Multi-Scale Analysis (MSA), by von Dreifus and Klein who developed and cleared ideas from a host of earlier papers.

For $d > 1$ and small values of g, one expects an absolutely continuous component to appear in the spectrum of $\mathsf{H}^{(1)}(\omega)$ (the Anderson-Mott *phase transition*).

For $N > 1$ particles the situation was obscurted by the above-mentioned difficulties, until Chulaevsky and myself proved the following assertion:

Take a N-particle random Hamiltonian in $\ell_2(\mathbb{Z}^{d\times N})$:

$$\mathbf{H}^{(N)}(\omega)f(\mathbf{x})$$

$$= -\frac{1}{2}\sum_{1\leq i\leq N}\sum_{1\leq j\leq d}\Big[f(\mathbf{x}+\mathbf{e}_i^j)+f(\mathbf{x}-\mathbf{e}_i^j)-2f(\mathbf{x})\Big]$$

$$+\sum_{1\leq i_1<i_2\leq N}\Phi(|x_{i_1}-x_{i_2}|)f(\mathbf{x})+g\sum_{1\leq i\leq N}V(x_i;\omega)f(\mathbf{x}),$$

$$\mathbf{x}=(x_1,\dots,x_N)\in\mathbb{Z}^{d\times N}.$$

Under the above conditions on $V(x;\omega)$, $x\in\mathbb{Z}^d$, take $|g|\geq g^*$. Then, $\mathbb{P}$-a.s., $\mathbf{H}^{(N)}(\omega)$ has a pure point spectrum.

Moreover, $\exists$ *a non-random* $m^* > 0$ *such that the eigenfunctions* $\mathbf{\Psi}_j$ *of* $\mathbf{H}^{(N)}(\omega)$ *obey*

$$\left|\mathbf{\Psi}_j(\mathbf{x};\omega)\right| \leq c_j(\omega)\mathrm{e}^{-m^*|\mathbf{x}|},\ \ \mathbf{x} \in \mathbb{Z}^{d\times N}.$$

See the papers by V.Chulaevsky & YS:

Wegner bounds in a two-particle Anderson tight-binding model.*Comm.Math.Phys.*,**283**(2008),479;

Eigenfunctions in a two-particle Anderson tight-binding model. *Ibid.*,**289** (2009),701;

Mutlti-particle Anderson localisation: induction in the number of particles.*Math.Phys.Anal.Geom.*,**12**(2009),117.

This result was quickly surpassed in

M. Aizenman, S. Warzel. Localization bounds for multi-particle systems. *Comm. Math. Phys.*, **293** (2009), 903.

Here, A & W consider a ‘dynamic’ localisation: a concept involving decay of unitary matrix elements

$$\left|\left\langle \delta_{\mathbf{x}}, \exp\ \mathrm{i}\,t\,\mathbf{H}^{(N)}\delta_{\mathbf{y}}\right\rangle\right| \leq C(\omega)\mathrm{e}^{-m^*|\mathbf{x}-\mathbf{y}|},\ \mathbf{x},\mathbf{y}\in\mathbb{Z}^{d\times N}.$$

By using the so-called *Fractional Moment Method* (FMM), this holds uniformly in $t\in\mathbb{R}$.

This implies the above result by C & S.

Altogether, it seems that a road is now open towards a ‘genuine’ Anderson–Mott theory, with a positive density of particles. More precisely, set

$$\mathbf{B}_L = \Big([-L,L]^d \cap \mathbb{Z}^d\Big)^{\times N} \text{ (an N-particle box),}$$

consider operator $\mathbf{H}^{(N)}_{\mathbf{B}_L}$ in $\ell_2\big(\mathbf{B}_L)$ and let

$$N, L \to +\infty \ \text{ with } \ \frac{N}{(2L+1)^d} \ \to \ \rho > 0.$$

The aim is to control localisation length $1/m^*$, in one way (MSA) or another (FMM) in such a limit.

However, this seems to be a hard problem.

In such a situation, Maths historically would look at Physics for inspiration. There is a physical paper D.M.Basco,I.L.Aleiner,B.L.Altshuler. Metal-insulator transition in a weakly interacting many-electron system with localized single-particle states. *Ann. Phys.*,**321**(2006),1126.

It claims that, for large disorder, a weak interaction doesn't change the qualitative localisation picture as compared to a non-interacting system.

Such a claim is convincing in some part, but not entirely.

And to a cetrtain extent the BAA claim is not in absolute harmony with an earlier paper

D.L. Shepelyansky. Three-dimensional Anderson transition for two electrons in two dimensions. *Phys. Rev. B*, **61** (2000), 4588.

Here a long-range repulsive (Culomb-type) interaction is credited with a ‘delocalisation’ phenomenon where the localisation length $1/m^*$ becomes infinite.

In short, we mathematicians have been left alone with such an important business: a rare occasion.

And so we try to proceed without a physical cover.

For a continuous space single-particle model in $\mathbb{R}^d$,

the Hamiltonian looks symbolically the same as on $\mathbb{Z}^d$:

$$\mathsf{H}^{(1)}(\omega) = -\frac{1}{2}\Delta + V(\,\cdot\,;\omega),$$

but has $\Delta = \sum_{1\le j\le d} \partial^2/\partial(\mathrm{x}^j)^2$, $x = (x^1, \dots, x^d) \in \mathbb{R}^d$.

Here, assuming that $V(x;\omega) \ge 0$, one proves

exponential localisation in an interval $[0, E^*]$ with

$E^* > 0$. (A long list of authors contributed.)

The phase transition is expected in this model

when you look at larger values of energy E.

For an N-particle system in a continuous space

with $\mathbf{x} = (x_1, \ldots, x_N) \in \mathbb{R}^{d\times N}$, the Hamiltonian reads

$$\mathbf{H}^{(N)} f(\mathbf{x}) = -\frac{1}{2}\boldsymbol{\Delta} f(\mathbf{x})$$

$$+ \sum_{1\leq i_1<i_2\leq N} \Phi(|x_{i_1} - x_{i_2}|) f(\mathbf{x}) + \sum_{1\leq i\leq N} V(x_i) f(\mathbf{x}).$$

Here, *assuming that* $V(x;\omega) \geq 0$, *exponential localisation holds again in an interval* $[E^0, E^0 + E^*]$. *Here* $E^* > 0$ *is a small number while* E^0 *is the lower end of the spectrum for the (non-random) operator*

$$-\frac{1}{2}\boldsymbol{\Delta} f(\mathbf{x}) + \sum_{1\leq i_1<i_2\leq N} \Phi(|x_{i_1} - x_{i_2}|) f(\mathbf{x}).$$

See: A.Boutet de Monvel, V.Chulaevsky, P.Stollmann, in preparation.