Example Sheet 2

1. The rooted binary tree is an infinite graph T with one distinguished vertex R from which comes a single edge; at every other vertex there are three edges and there are no closed loops. The random walk on T jumps from a vertex along each available edge with equal probability. Show that the random walk is transient.

2. Show that for the Markov chain $(X_n)_{n\geq 0}$ in Example 12 from Sheet 1

$$\mathbb{P}(X_n \to \infty \text{ as } n \to \infty) = 1\,.$$

Suppose the transition probabilities satisfy instead

$$p_{ii+1} = \left(\frac{i+1}{i}\right)^{\alpha} p_{ii-1}$$

for some $\alpha \in (0, \infty)$. What then is the value of $\mathbb{P}(X_n \to \infty \text{ as } n \to \infty)$?

3. Show that the simple symmetric random walk in $\mathbb{Z}^4$ is transient.

4. Find all invariant distributions of the transition matrix

$$P = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

5. Gas molecules move about randomly in a box which is divided into two halves symmetrically by a partition. A hole is made in the partition. Suppose there are N molecules in the box. Show that the number of molecules on one side of the partition just after a molecule has passed through the hole evolves as a Markov chain. What are the transition probabilities? What is the invariant distribution of this chain?

6. A particle moves on the eight vertices of a cube in the following way: at each step the particle is equally likely to move to each of the three adjacent vertices, independently of its past motion. Let i be the initial vertex occupied by the particle, o the vertex opposite i. Calculate each of the following quantities:

(i) the expected number of steps until the particle returns to i;
(ii) the expected number of visits to o until the first return to i;
(iii) the expected number of steps until the first visit to o.

7. Find the invariant distributions of the transition matrices in Example 7 from Sheet 1, parts (a), (b) and (c), and use them to check your answers.

8. A fair die is thrown repeatedly. Let X_n denote the sum of the first n throws. Find

$$\lim_{n \to \infty} \mathbb{P}(X_n \text{ is a multiple of } 13).$$

9. Each morning a student takes one of the three books she owns from her shelf. The probability that she chooses book i is $\alpha_i, 0 < \alpha_i < 1, i = 1, 2, 3$, and choices on successive days are independent. In the evening she replaces the book at the left-hand end of the shelf. If p_n denotes the probability that on day n the student finds the books in the order 1,2,3, from left to right, show that, irrespective of the initial arrangement of the books, p_n converges as $n \to \infty$, and determine the limit.

10. In each of the following cases determine whether the stochastic matrix P is reversible:

(a) $\begin{pmatrix} p & 1-p \\ q & 1-q \end{pmatrix}$; (b) $\begin{pmatrix} 0 & p & 1-p \\ 1-p & 0 & p \\ p & 1-p & 0 \end{pmatrix}$;

(c) $I = \{0, 1, \ldots, N\}$ and $p_{ij} = 0$ if $|j - i| \geq 2$;

(d) $I = \{0, 1, 2, \ldots\}$ and $p_{01} = 1$, $p_{ii+1} = p$, $p_{ii-1} = 1 - p$ for $i \geq 1$.

11. A professor has N umbrellas, which he keeps either at home or in his office. He walks to and from his office each day, and takes an umbrella with him if and only if it is raining. Throughout each journey, it either rains, with probability p, or remains fine, with probability $1 - p$, independently of the past weather. What is the long run proportion of journeys on which he gets wet?

12. Consider a pack of cards labelled 1,2, .., 52. We repeatedly take the top card and insert it uniformly at random in one of the 52 possible places, that is, either on the top or on the bottom or in one of the 50 places inside the pack. How long on average will it take for the bottom card to reach the top?

Let p_n denote the probability that after n iterations the cards are found in increasing order. Argue that, irrespective of the initial ordering, p_n converges as $n \to \infty$, and determine the limit p. **Hint:** *Do not miss the property of aperiodicity.*

Show that, at least until the bottom card reaches the top, the ordering of cards inserted beneath it is uniformly random. Hence or otherwise show that, for all n,

$$|p_n - p| \leq 52(1 + \log\ 52)/n.$$

Hint: *The Chebyshev inequality* $\mathbb{P}(Y > n) \leq \mathbb{E}Y/n$ *may be helpful.*

13. In chess, a queen can moves in any direction (horizontal, vertical, and two diagonal). Suppose the queen moves at random around the chess board, choosing a new square with equal probability from the squares it can reach in one move. Suppose she starts at the bottom left corner and let X_n be the queen position at time n.

(i) Show that (X_n) is a Markov chain and describe its states and transition probabilities.

(ii) From the detailed balance equations, or otherwise, determine the equilibrium probabilities for the chain.

(iii) What is the expected number of moves the queen will make before first returning to its starting point?

The chess board is 8×8*: you may label its squares by pairs* (i, j) *where* $i, j = 1, \ldots, 8$*. The number of moves the queen can make depends of course on the square: from each of the 4 central squares she can make* 27 *moves, whereas from each of the border squares she can make only* 21 *moves.*