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MARKOV CHAINS

5. Recurrence and transience

Recurrence and transience; equivalence of transience and summability of n-step transition probabilities; equivalence of recurrence and certainty of return. Recurrence as a class property, relation with closed classes. Simple random walks in dimensions one, two and three.

Given a state $i \in I$, we call it *recurrent* (R) if

$$\mathbb{P}_i\big(X_n = i \text{ for infinitely many } n\big) = 1, \quad \Leftrightarrow \quad \text{the series } \sum_{n\geq 0} p_{ii}^{(n)} \text{ diverges} \tag{5.1.1}$$

and *transient* (T) if

$$\mathbb{P}_i\big(X_n = i \text{ for infinitely many } n\big) = 0, \quad \Leftrightarrow \quad \text{the series } \sum_{n\geq 0} p_{ii}^{(n)} \text{ converges.} \tag{5.1.2}$$

The last condition means that

$$\mathbb{P}_i\big(X_n = i \text{ for finitely many } n\big) = 1.$$

(The symbol $\Leftrightarrow$ means equivalence.) Note that all probabilities involved take values 0 or 1 but not from $(0,1)$. The equivalence of properties listed in (5.1.1) and that of properties listed in (5.1.2) is the subject of Theorem 5.1 below.

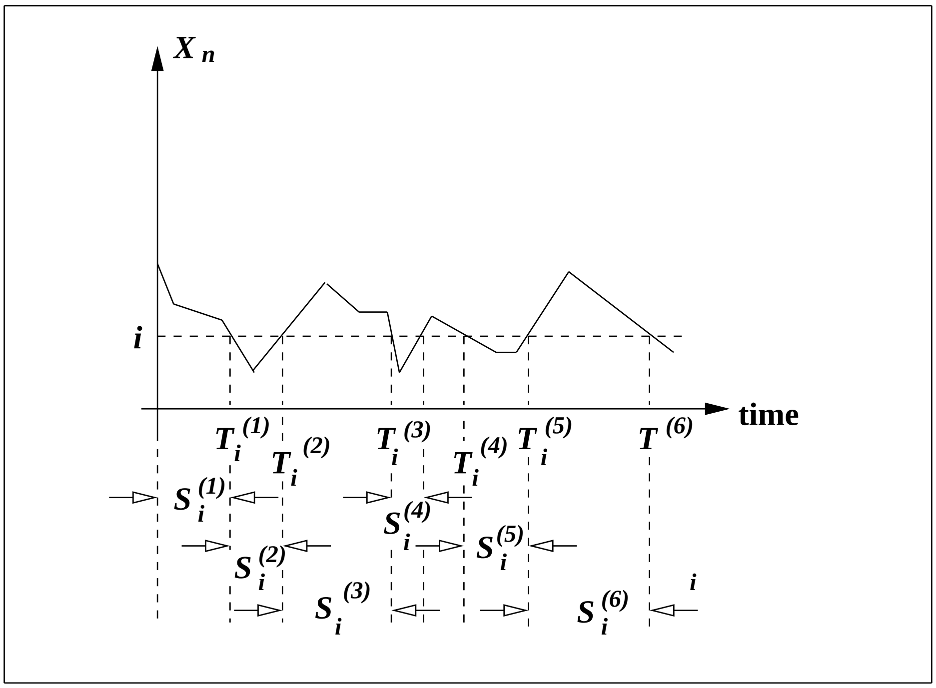
Given $r = 0, 1, \ldots$, define the rth *passage time* $T_i^{(r)}$ to state i by $T_i^{(0)} = 0$ and

$$T_i^{(1)} = \inf\,\big[n \geq 1 : X_n = i\big],\; T_i^{(r+1)} = \inf\,\big[n \geq T_i^{(r)} + 1 : X_n = i\big], r \geq 1, \tag{5.2}$$

and the *length* of the rth *excursion* to i by

$$S_i^{(r)} = \begin{cases} T_i^{(r)} - T_i^{(r-1)}, & \text{if } T_i^{(r-1)} < \infty, \\ 0, & \text{if } T_i^{(r-1)} = \infty. \end{cases} \tag{5.3}$$

When $r = 1$, we simply write $T_i^{(1)} = T_i$; in addition, $T_i^{(r+1)}$ is set to equal $+\infty$ when $T_i^{(r)} = +\infty$. See the diagram below.



Next, define the return probability to state i:

$$f_i := \mathbb{P}_i(T_i < +\infty). \tag{5.4}$$

This probability may be equal to or less than than one; in Theorem 5.1 it will be proved that

$$\text{state } i \text{ is R} \quad \text{iff} \quad f_i = 1 \tag{5.5.1}$$

and

$$\text{state } i \text{ is T} \quad \text{iff} \quad f_i < 1 \tag{5.5.2}$$

Therefore, every state is either R or T. In Theorem 5.2 we will show that recurrence or transience are class properties: two states from the same class are either both R or both T.

Before we proceed further, let us analyse the structure of the series $\sum_{n\geq 0} p_{ii}^{(n)}$ in (5.1.1) and (5.1.2):

$$\begin{aligned}\sum_{n\geq 0} p_{ii}^{(n)} &= \sum_n \mathbb{P}_i(X_n = i) = \sum_n \mathbb{E}_i \mathbf{1}(X_n = i) \\ &= \mathbb{E}_i \sum_n \mathbf{1}(X_n = i) = \mathbb{E}_i V_i\end{aligned} \tag{5.6}$$

where

$$V_i \ = \ \# \text{ of visits to } i \ = \ \text{time spent in } i \ = \ \sum_{n\geq 0} \mathbf{1}(X_n = i). \tag{5.7}$$

Then equations (5.1.1) and (5.1.2) can be written as

$$\mathbb{P}_i\left(X_n = i \text{ for } \begin{cases}\text{infinitely} \\ \text{finitely}\end{cases} \text{many times}\right) = 1 \Leftrightarrow \mathbb{E}_i V_i \begin{cases} = +\infty, \\ < +\infty. \end{cases} \tag{5.8}$$

We see that, for a Markov chain,

$$\mathbb{P}_i\left(V_i \begin{cases} = +\infty \\ < +\infty \end{cases} \right) \quad \Leftrightarrow \quad \mathbb{E}_i V_i \begin{cases} = +\infty \\ < +\infty \end{cases} \tag{5.9}$$

and the case where $\mathbb{P}_i(V_i < +\infty) = 1$ but $\mathbb{E}_i V_i = +\infty$ does not occur.

Theorem 5.1 *In a (λ, P)-Markov chain, $\forall$ state $i \in I$ we have the dichotomy:*

$$\begin{aligned}&\sum_{n\geq 0} p_{ii}^{(n)} \\ &= \mathbb{E}_i V_i \begin{cases} = +\infty \Leftrightarrow \mathbb{P}_i(V_i = +\infty) = 1 \Leftrightarrow f_i = \mathbb{P}_i(T_i < +\infty) = 1 : \text{R} \\ < +\infty \Leftrightarrow \mathbb{P}_i(V_i < +\infty) = 1 \Leftrightarrow f_i = \mathbb{P}_i(T_i < +\infty) < 1 : \text{T} \end{cases}\end{aligned} \tag{5.10}$$

Therefore, every state is either recurrent or transient.

Proof The random variable T_i is a stopping time. By the strong Markov property,

$$\mathbb{P}_i(V_i \geq 2) = \mathbb{P}_i\left(X_n = i \text{ for at least two values of } n \geq 1\right) = f_i^2, \tag{5.10}$$

and more generally, $\forall\ k$

$$\mathbb{P}(V_i \geq k) = \mathbb{P}_i\left(X_n = i \text{ for at least } k \text{ values of } n \geq 1\right) = f_i^k. \tag{5.11}$$

Let us highlight the event under consideration:

$$B_k^{(i)} := \left\{V_i \geq k\right\} = \left\{X_n = i \text{ for at least } k \text{ values of } n \geq 1\right\}.$$

Then, obviously, events $B_k^{(i)}$ are decreasing with k: $B_1^{(i)} \supseteq B_2^{(i)} \supseteq \ldots$, and the event

$$\left\{V_i = +\infty\right\} = \left\{X_n = i \text{ for infinitely many values of } n\right\}$$

is the intersection $\cap_{k\geq 1} B_k^{(i)}$. Hence,

$$\begin{aligned} \mathbb{P}_i(V_i = +\infty) &= \mathbb{P}_i\big(X_n = i \text{ for infinitely many } n\big) \\ &= \mathbb{P}_i\left(\cap_{k\geq 1} B_k^{(i)}\right) \\ &= \lim_{k\to\infty} \mathbb{P}(B_k^{(i)}) = \lim_{k\to\infty} f_i^k = \begin{cases} 1, & f_i = 1 \\ 0, & f_i < 1. \end{cases} \end{aligned} \tag{5.12}$$

Next,

$$\begin{aligned} \sum_{n\geq 0} p_{ii}^{(n)} &= \mathbb{E}_i V_i = \sum_{r\geq 0} \mathbb{P}_i(V_i \geq r) \\ &= \sum_{r\geq 0} f_i^r \begin{cases} = +\infty, & f_i = 1, \text{ i.e., } \mathbb{P}_i(V_i = +\infty) = 1, \\ < +\infty, & f_i < 1, \text{ i.e., } \mathbb{P}_i(V_i = +\infty) = 0. \end{cases} \end{aligned} \tag{5.13}$$

QED

Theorem 5.1 will be repeatedly used in the analysis of recurrence and transience of states of various chains.

An alternative proof of Theorem 5.1 exploits the probability generating functions of random variable T_i. Set

$$\begin{aligned} f_i(n) &= \mathbb{P}_i(T_i = n) \\ &= \mathbb{P}_i\left(X_n = i \text{ but } X_l \neq i \text{ for } l = 1, \ldots, n-1\right),\ n \geq 1, \end{aligned} \tag{5.14}$$

and

$$F(z)(= F_i(z)) = \mathbb{E}z^{T_i} = \sum_{n\geq 1} z^n f_i(n), \quad |z| < 1, \tag{5.15}$$

then $f_i = \lim_{z\to 1} F(z)$.

On the other hand,

$$p_{ii}^{(n)} = \mathbb{P}_i(X_n = i) = f_i(n) + f_i(n-1)p_{ii} + \ldots + f_i(1)p_{ii}^{(n-1)}; \tag{5.16}$$

this implies that if

$$U(z)(= U_i(z)) = \sum_{n\geq 1} p_{ii}^{(n)} z^n, \quad |z| < 1, \tag{5.17}$$

then

$$U(z) = F(z) + F(z)U(z), \ \text{ i.e. } U(z) = \frac{F(z)}{1 - F(z)}.$$

Hence, the limiting value $\lim_{z\to 1} U(z)$ is finite if and only if $\lim_{z\to 1} F(z) < 1$. That is, (5.13) holds true if and only if $f_i < 1$.

Remark 5.1 We established that state i is recurrent if and only if $\mathbb{P}_i(T_i < \infty) = 1$, i.e. the return time to i is finite with probability 1. However, the mean $\mathbb{E}_i T_i$ can be finite ot infinite. This divides recurrent states into two distinct categories: positive recurrent and null recurrent (see later).

For convenience we repeat the definition of a communicating class:

Definition 5.2 States $i, j \in I$ belong to the same *communicating class* if $p_{ij}^{(n)} > 0$ and $p_{ji}^{(n')} > 0$ for some $n, n' \geq 0$. The communicating classes form a partition of the state space I, when I is countable, some of them may be infinite, and the number of communicating classes can also be infinite. Next, a communicating class C is called *closed* if $\forall\ i \in C$, if $i \to j$ then $j \in C$. Finally, we say that the chain is *irreducible* if it has a unique communicating class (automatically closed). In other words, in an irreducible chain, the whole of the state space I is a single (closed) communicating class.

Remark 5.2 Observe that if the state space I is finite, the definition of a transient state coincides with that of a non-essential state (i.e., a state from a non-closed communicating class). In other words, in the finite case every state from a non-closed class is transient, and every state from a closed class is recurrent. However, as we noted in Remark 2.1, in the case of a countable DTMC a closed class can consist entirely of transient states, which are, from a 'physical' point of view, non-essential. It shows that in the countable case the concept of transience is more relevant than that of a closed communicating class.

Our aim now is to prove that recurrence and transience are class properties. This means that if states i, j lie in the same communicating class then they are either both recurrent or both transient. We therefore could use

Definition 5.3 A communicating class is called *recurrent* (*transient*) if all its states are recurrent (respectively, transient).

Theorem 5.2 *Within the same communicating class, all states are of the same type. Every finite closed communicating class is recurrent.*

Proof Let C be a communicating class. Then, $\forall$ distinct $i, j \in C$, $p_{ij}^{(m)} > 0$ and $p_{ji}^{(n)} > 0$ for some $m, n \geq 1$. Then $\forall\ r \geq 0$:

$$p_{ii}^{(n+m+r)} \geq p_{ij}^{(m)} p_{jj}^{(r)} p_{ji}^{(n)} \ \text{ and } \ p_{jj}^{(n+m+r)} \geq p_{ji}^{(n)} p_{ii}^{(r)} p_{ij}^{(m)}, \tag{5.18}$$

as the RHS in each inequality takes into account only a part of the possibilities of return.

Hence

$$p_{jj}^{(r)} \le \frac{p_{ii}^{(n+m+r)}}{p_{ij}^{(m)} p_{ji}^{(n)}} \tag{5.19.1}$$

and, for $r \ge n+m$,

$$p_{jj}^{(r)} \ge p_{ji}^{(n)} p_{ii}^{(r-n-m)} p_{ij}^{(m)}. \tag{5.19.2}$$

Then the series $\sum\limits_r p_{ii}^{(r)}$ and $\sum\limits_r p_{jj}^{(r)}$ converge or diverge together.

Now let C be a finite closed communicating class, and $j \in C$. Then, with $X_0 = j \in C$, $X_n \in C \ \forall\ n$. Hence, $\exists$ state $i \in C$ visited infinitely often, with $\mathbb{P}_j(V_i = \infty) > 0$. (In other words, it cannot be that $\mathbb{P}_j(V_i = \infty) = 0$ for all $i \in C$: we run the chain for infinitely many times among finitely many states, implying that $\sum\limits_{i \in C} \mathbb{P}_j(V_i = \infty) = 1$.) By the Strong Markov property,

$$\mathbb{P}_j(V_i = \infty) = \mathbb{P}_j(T_i < \infty)\, \mathbb{P}_i(V_i = \infty).$$

Then the probability $\mathbb{P}_i(V_i = \infty)$ cannot be 0 (and hence should be 1). Thus, state i is recurrent. Then every state from C is recurrent. QED

Definition 5.2. A transition matrix P (and a (λ, P) Markov chain) is called recurrent (transient) if every state i is recurrent (respectively, transient).

We conclude this section with one more statement involving passage, or return, times.

Theorem 5.3 (Non-examinable) *If P is irreducible and recurrent then each random variable T_j (the passage time to state j) is finite with probability* 1. *That is, $\mathbb{P}(T_j < \infty) = 1 \ \forall\ j$ and initial distribution λ.*

Proof By the Markov property

$$\mathbb{P}(T_j < \infty) = \sum_i \lambda_i \mathbb{P}_i(T_j < \infty).$$

Given i, take m with $p_{ji}^{(m)} > 0$. Write

$$1 = \mathbb{P}_j\big(V_j = \infty\big) \leq \mathbb{P}_j\big(X_n = j \text{ for some } n \geq m\big)$$

(obviously, there is equality here, but the inequality will also do). Further,

$$\begin{aligned}
&\mathbb{P}_j\big(X_n = j \text{ for some } n \geq m\big) \\
&= \sum_k p_{jk}^{(m)} \mathbb{P}_j\big(X_n = j \text{ for some } n \geq m \big| X_m = k\big) \\
&= \sum_k p_{jk}^{(m)} \mathbb{P}_k\big(T_j < \infty\big) \leq \sum_k p_{jk}^{(m)} = 1.
\end{aligned}$$

We see that each summand $p_{jk}^{(m)} \mathbb{P}_k\big(T_j < \infty\big)$ must be equal to $p_{jk}^{(m)}$; otherwise (i.e. when $p_{jk}^{(m)} \mathbb{P}_k\big(T_j < \infty\big) < p_{jk}^{(m)}$ for some k) we would have that $1 < 1$. Therefore,

$$\mathbb{P}_i\big(T_j < \infty\big) p_{ji}^{(m)} = p_{ji}^{(m)}, \quad \text{i.e.} \quad \mathbb{P}_i\big(T_j < \infty\big) = 1.$$

This is true $\forall\ i$, hence $\forall$ initial distribution λ. Also, it is true $\forall\ j$. QED

Example 5.1 (Math Tripos, Markov Chains, Part IB, 2004, 111H) Let $P = (p_{ij})$ be a transition matrix. What does it mean to say that P is *(a)* irreducible, *(b)* recurrent?

Suppose that P is irreducible and recurrent and that the state space contains at least two states. Define a new transition matrix $\widehat{P} = (\widehat{p}_{ij})$ by

$$\widehat{p}_{ij} = \begin{cases} 0 & \text{if } i = j, \\ (1 - p_{ii})^{-1} p_{ij} & \text{if } i \neq j. \end{cases}$$

Prove that $\widehat{P}$ is also irreducible and recurrent.

Solution A Markov chain is called irreducible if it has a single communicating class. Equivalently, every pair of states i, j communicate; that is, the n-step transition probability $p_{ij}^{(n)} > 0$ for some $n (= n(i,j)) \geq 1$. Another equivalent condition is that $\exists$ a finite sequence of non-repeated states $i = i_0$, i_1, ..., $i_m = j$ with $p_{i_l i_{l+1}} > 0$. A chain is called recurrent if $\forall$ state i:

$$\mathbb{P}_i\big(X_n = i \text{ for infinitely many } n\big) = 1,$$

or equivalently,

$$\mathbb{P}_i\big(X_n = i \text{ for some } n \geq 1\big) = 1.$$

When the chain is irreducible, it is enough to check that $\exists$ a state i with the above property. Also, if the chain is irreducible, it is recurrent if and only if $\forall$ state i, the minimal non-negative solution to the equation $hP = h$ satisfying $h_i = 1$ has $h_j = 1 \ \forall \ j$.

If $P = (p_{ij})$ is irreducible then $p_{ii} < 1 \ \forall$ state i (unless the total number of states is one). Matrix $\widehat{P}$ describes the Markov chain obtained from the original DTMC by recording the jumps to the new state only; clearly it is irreducible. Formally, take the sequence i_0, ..., i_m as above, then $\widehat{p}_{i_l i_{l+1}} > 0$. Now check the recurrence of $\widehat{P}$: if in the original chain $p_{ii} = 0$ then the return to state i occurs in both chains on the same event, hence the return probability to state i will be the same. If $p_{ii} > 0$ then in the new chain, the return probability is equal to

$$\frac{1}{1-p_{ii}}\mathbb{P}_i(\text{return to } i \text{ after time 1 in the original chain})$$

$$= \frac{1}{1-p_{ii}}(1-p_{ii})$$

which is 1. Alternatively, $h\widehat{P} = h$ if and only if $hP = h$, i.e. the solutions to both equations are the same. Hence, the minimal solution to $h\widehat{P} = h$ with $h_i = 1$ is the same as that to $hP = h$. Therefore, it is $\equiv 1$, and the new chain is recurrent if and only if the original one is.

Example 5.2 Consider a homogeneous birth and death process, where

$$p_{ii+1} = p, \ p_{ii-1} = 1-p, \ i \geq 1, \ \ p_{01} = r, \ p_{00} = 1-r.$$

It is often called a homogeneous *random walk* on the set of non-negative integers $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$ with a barrier at 0 (absorbing when $r = 0$, reflecting when $r = 1$, or a random combination of the two when $r \in (0, 1)$).

Case 1: $0 < r \leq 1$ and $0 < p \leq 1/2$. Here we have a single communication class which is closed. That is, the chain is irreducible. For state $i = 0$: the return probability is

$$\begin{aligned} f_0 &= \mathbb{P}_0(T_0 < +\infty) \\ &= 1 - r + r \cdot \mathbb{P}_1\big(H^{(0)} < +\infty\big) \ \text{ conditional on the 1st step} \qquad (5.20) \\ &= 1 - r + r \cdot h_1 = 1 - r + r = 1: \ \text{ state 0 is R.} \end{aligned}$$

Hence, all states are R, and the whole process is R.

Case 2: $0 < r \leq 1$ and $1/2 < p < 1$. Again, the chain is irreducible. The return probability to state $i = 0$:

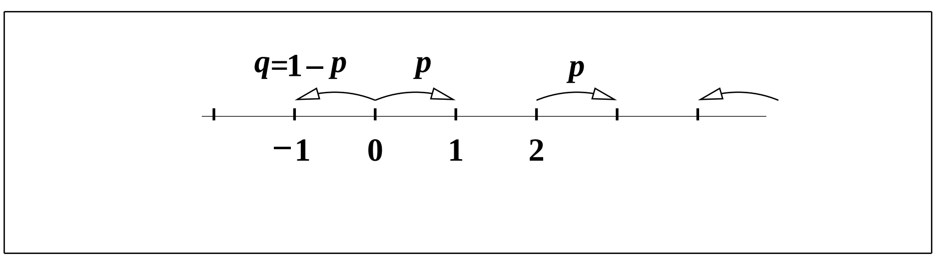
$$f_0 = 1 - r + r \cdot h_1 = 1 - r + r\frac{1-p}{p} < 1 - r + r = 1 : \quad \text{state 0 is T.} \quad (5.21)$$

The remaining cases are non-generic: either p or r equal 1 or 0.

For brevity, consider just one of them:

Case 3: $0 \leq r \leq 1$ and $p = 0$: from state $i \geq 1$ the process always jumps to $i - 1$ while from 0 it can jump to 1 when $0 < r \leq 1$ and back to 0 when $0 \leq r < 1$ (when $0 < r < 1$, both possibilities occur). The chain is reducible: every state $i > 1$ forms an open CC and so is T. For $0 < r \leq 1$: the pair $\{0, 1\}$ forms a closed CC, and both $i = 0$ and $i = 1$ are R-states. For $r = 0$ state $i = 1$ is T and state $i = 0$ R.

Example 5.3 (Random walks on lattices) Random walks on cubic lattices are popular and interesting models of countable Markov chains. Here we have a 'particle' that jumps at times $n = 1, 2, \ldots$ from its current position $\underline{i} \in \mathbb{Z}^d$ to another site $\underline{j} \in \mathbb{Z}^d$ with probability $p_{\underline{i}\underline{j}}$, regardless of the past trajectory. We will mostly focus on homogeneous nearest-neighbour random walks where probabilities $p_{\underline{i}\underline{j}}$ are > 0 only when $\underline{i}$ and $\underline{j}$ are neighbouring sites and depend only on the direction from $\underline{i}$ to $\underline{j}$ (i.e. are determined by $p_{\underline{0},\underline{j}}$ where $\underline{j}$ is a neighbour of the origin $\underline{0} = (0, \ldots, 0)$). For $d = 1$ lattice $\mathbb{Z}^d$ is simply the set of integers; a random walk here is specified by probabilities p and $q = 1 - p$ of jumps to the right and the left.



This is an intuitively appealing extended version of the drunkard model (or birth-and-death process); see Example 2.1. Here, the state space is $I = \mathbb{Z}(= \mathbb{Z}^1)$, and the transition probability matrix is infinite and has a distinct 'diagonal' structure

$$P = \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & q & 0 & p & 0 & \dots & \ddots \\ \ddots & 0 & q & 0 & p & 0 & \ddots \\ \ddots & \ddots & 0 & q & 0 & p & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}, \tag{5.22}$$

with entries p above and q below the main diagonal, and the rest filled with zeros.

The probability $p_{00}^{(n)}$ is obviously zero when n is odd (you can't return after an odd number of jumps). If $n = 2k$ is even then the return is equivalent to have exactly k jumps to the right and k jumps to the left. Hence,

$$p_{00}^{(2k)} = \frac{(2k)!}{k!k!} p^k q^k.$$

For k large, by Stirling formula $n! \sim \sqrt{2\pi n} n^n e^{-n}$,

$$\frac{(2k)!}{k!k!} =\sim \frac{\sqrt{2\pi \times 2k}(2k)^{2k} e^{-2k}}{2\pi k k^{2k} e^{-2k}} = \frac{1}{\sqrt{\pi k}} 4^k.$$

Hence, the series $\sum\limits_n p^{(n)} = \sum\limits_k p_{00}^{(2k)}$ converges or diverges together with

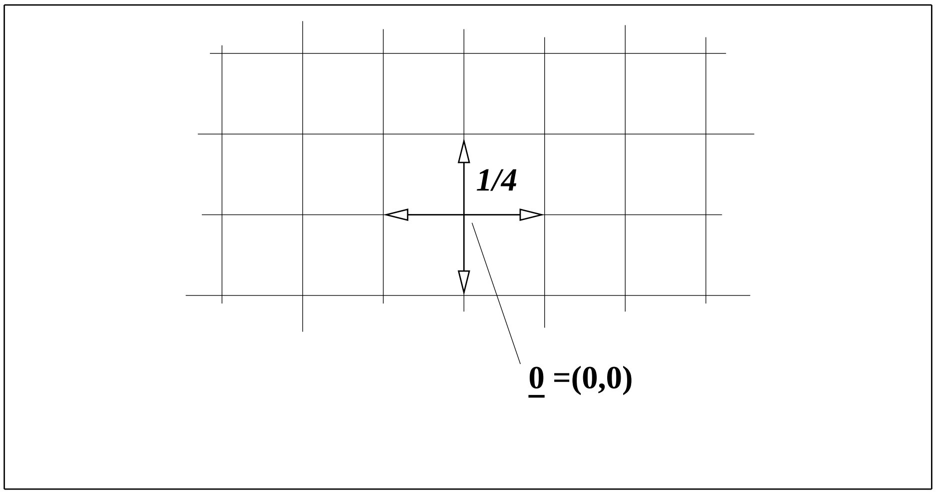
$$\sum_k \frac{1}{\sqrt{k}} \big(4p(1-p)\big)^k.$$

Now, the parabola $p \mapsto p(1-p)$ reaches its maximum on $[0,1]$ at $p = 1/2$; the maximal value equals $1/4$. Hence,

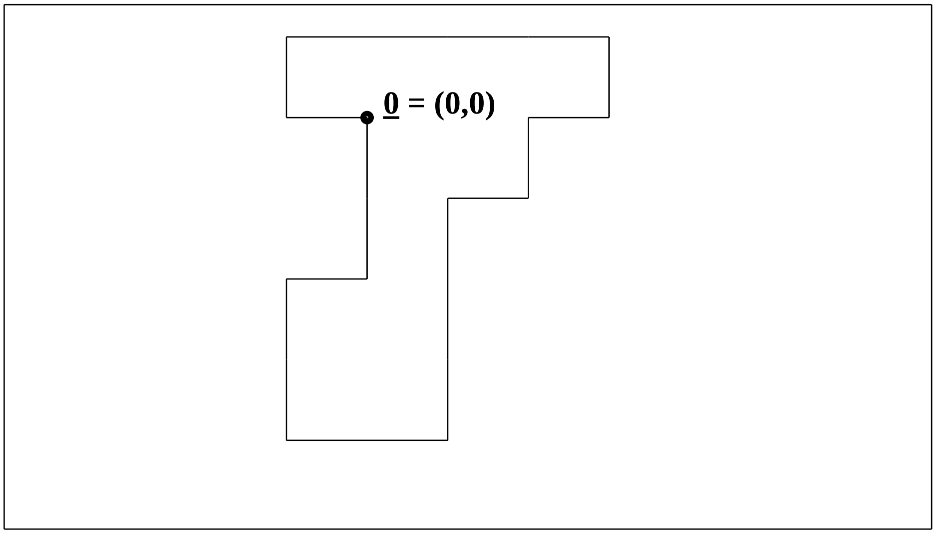
$$4p(1-p) \leq 1, \quad \text{with equality iff} \quad p = \frac{1}{2}.$$

Therefore the above series diverges when $p = 1/2$ and converges when $p \neq 1/2$. We conclude that for $d = 1$, state 0 is transient when $p \neq 1/2$ and recurrent when $p = 1/2$. As the model is homogeneous, the same holds for each state $i \in \mathbb{Z}$; one simply says that for $p = 1/2$ the one-dimensional nearest-neighbour random walk is recurrent and for $p \neq 1/2$ transient. The case $p = 1/2$ is called symmetric.

For $d = 2$, $\mathbb{Z}^2$ is a plane square lattice; here we will consider the symmetric nearest-neighbour random walk where probabilities to jump in every direction are the same and equal $1/4$.

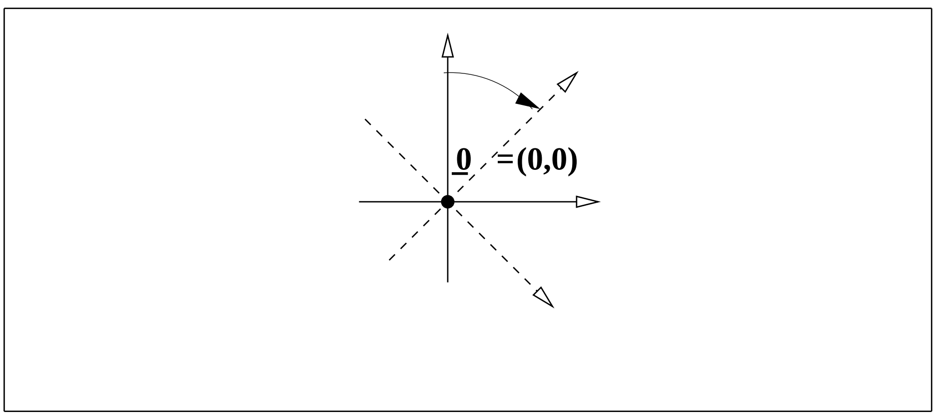


Every closed path on $\mathbb{Z}^2$ must have equally many jumps to the left and the right and equally many jumps up and down.



Hence again $p^{(n)}_{\underline{0}\underline{0}} = 0$ when n is odd.

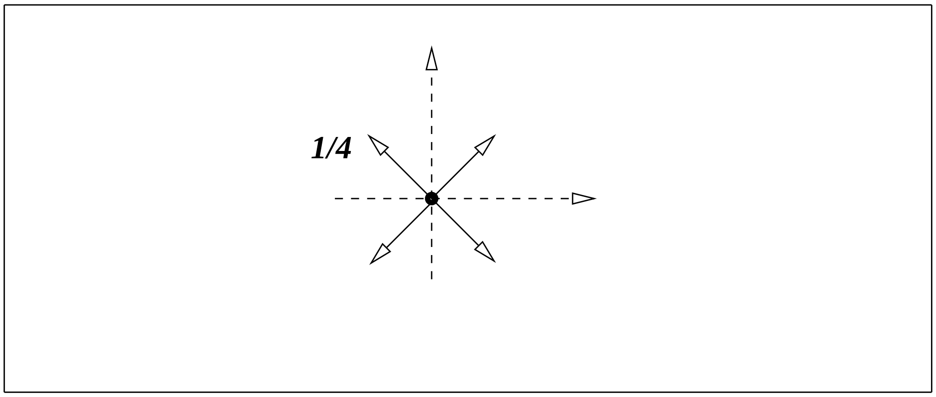
A useful idea is to project the random walk onto orthogonal axes rotated by $\pi/4$.



The moves are in one-to-one correspondence

old coordinates: chain (X_n)		new coordinates: chain (X'_n)
move by vector $\pm(1;0)$	$\leftrightarrow$	move by vector $\pm\,(1/\sqrt{2})(1;1)$
move by vector $\pm(0;1)$		move by vector $\pm\,(1/\sqrt{2})(-1;1)$.

In the new coordinates, up to a factor $\frac{1}{\sqrt{2}}$, the jumps are along diagonals of the unit square.

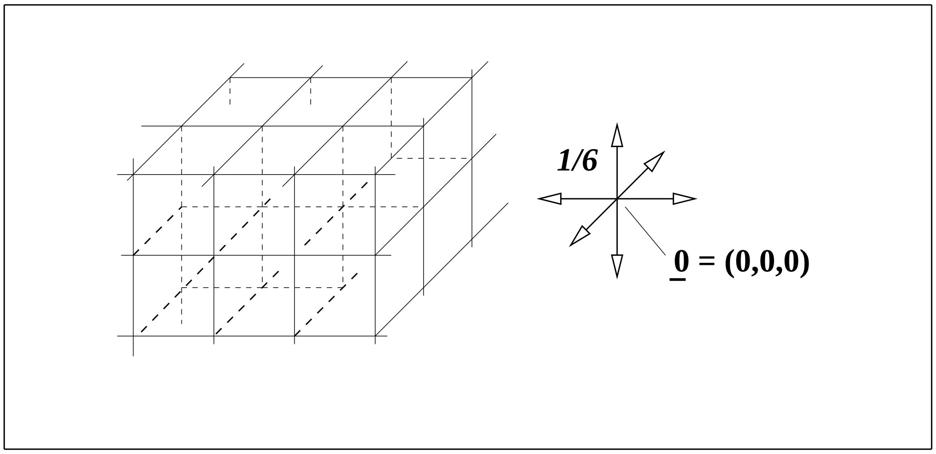


It means that the chain (X'_n) in the new coordinates is formed by a pair of independent symmetric nearest-neighbour random walks on $\mathbb{Z}$ (in the horizontal and vertical directions). Return to $\underline{0} = (0, 0)$ means return to 0 in each of them. Therefore, for $n = 2k$

$$p^{(2k)}_{\underline{00}} = \left(\frac{(2k)!}{k!k!} \frac{1}{2^{2k}} \right)^2 \approx \frac{1}{\pi k}. \tag{5.26}$$

Hence, $\sum_k p^{(2k)}_{\underline{00}} = \infty$, and the random walk is recurrent.

For $d = 3$, $\mathbb{Z}^3$ is the three-dimensional cubic lattice; we may think that it is an infinitely extended crystal. Then our walking particle may model a solitary quantum electron moving between heavy ions or atoms fixed at the sites of the lattice. The probability of moving to one of the six neighbours equals 1/6.



Still, $p_{\underline{0}\underline{0}}^{(n)} = 0$ when n is odd. If n is even, a path returns to $\underline{0} = (0,0,0)$ if and only if it makes equal numbers of jumps in each of three pairs of opposite directions (up/down, east/west, north/south). So,

$$p_{\underline{0}\underline{0}}^{(2k)} = \sum_{\substack{i,j,l \geq 0: \\ i+j+l=k}} \frac{(2k)!}{(i!)^2(j!)^2(l!)^2} \left(\frac{1}{6}\right)^{2k}$$

$$= \frac{(2k)!}{(k!)^2} \sum_{\substack{i,j,l \geq 0: \\ i+j+l=k}} \left(\frac{k!}{i!j!l!}\right)^2 \left(\frac{1}{6}\right)^{2k}$$

$$\leq \frac{(2k)!}{(k!)^2} \left(\max \, \frac{k!}{i!j!l!}\right) \frac{1}{3^k} \, \frac{1}{2^{2k}} \sum_{\substack{i,j,l \geq 0: \\ i+j+l=k}} \frac{k!}{i!j!l!} \, \frac{1}{3^k}.$$

Now, the sum

$$\sum_{\substack{i,j,l \geq 0: \\ i+j+l=k}} \frac{k!}{i!j!l!} = 3^k, \tag{5.27}$$

the number of ways to place k balls into 3 boxes. Also, for $k = 3m$

$$\frac{(3m)!}{m!m!m!} \geq \frac{(3m)!}{i!j!l!} \quad \text{whenever} \quad i+j+l = 3m. \tag{5.28}$$

In fact, suppose that $i < m < l$. Then when you pass to $i!j!l!$ from $(m!)^3$, you either (a) replace the 'tails' $(i+1)\dots m$ and $(j+1)\dots l$ of $m!$ by the product $(m+1)\dots(m+2m-i-j)$, i.e. $(m+1)\dots l$ when $j < m$ or (b) replace the tail $(i+1)\dots m$ by the product $(m+1)\dots j(m+1)\dots(3m-i-j)$, that is $(m+1)\dots j(m+1)\dots l$ when $j > m$. Either way you increase the denominator, hence decrease the ratio.

Then, for $n = 2k = 6m$:

$$p_{\underline{00}}^{(6m)} \leq \frac{(6m)!}{\big((3m)!\big)^2}\left(\frac{1}{2}\right)^{6m}\frac{(3m)!}{(m!)^3}\left(\frac{1}{3}\right)^{3m} \tag{5.29}$$

which, by Stirling, is

$$\approx \sqrt{2}\left(\frac{1}{\sqrt{2\pi}}\right)^3\frac{1}{m^{3/2}}. \tag{5.30}$$

Hence, $\sum\limits_m p_{\underline{00}}^{(6m)} < \infty$.

But for $m \geq 1$: $p_{\underline{00}}^{(6m)} \geq (1/6)^2 p_{\underline{00}}^{(6m-2)}$ and $p_{\underline{00}}^{(6m)} \geq (1/6)^4 p_{\underline{00}}^{(6m-4)}$, i.e.

$$p_{\underline{00}}^{(6m-2)} \leq 6^2 p_{\underline{00}}^{(6m)} \text{ and } p_{\underline{00}}^{(6m-4)} \leq 6^4 p_{\underline{00}}^{(6m)}.$$

Thus,

$$\sum_k p_{\underline{00}}^{(2k)} \leq \sum_m p_{\underline{00}}^{(6m)}(1 + 6^2 + 6^4) < \infty, \tag{5.31}$$

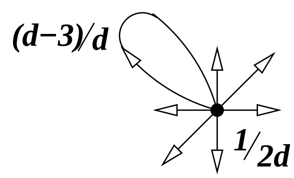
and the walk is transient.

A similar approach can be used in higher dimensions. But there is another way to establish transience in all dimensions $d > 3$. Namely, project the random walk (X_n^d) on $\mathbb{Z}^d$ to three dimensions by discarding all coordinates but the first three. The projected chain $\big(X_n^{\mathrm{proj}}\big)$ on $\mathbb{Z}^3$ stays where it is with probability $(d-3)/d$ (when the original walk jumps in one of the discarded directions), but when it jumps, it behaves as the nearest-neighbour symmetric walk in dimension 3

$$\mathbb{P}\left(X_{n+1}^{\mathrm{proj}} = \underline{i} \pm \underline{e}^\alpha \middle| X_n^{\mathrm{proj}} = \underline{i}\right) = \frac{1/(2d)}{1-(d-3)/d} = \frac{1}{6},\ \alpha = 1, 2, 3, \tag{5.32}$$

with

$$\underline{e}^1 = (1;0;0),\ \underline{e}^2 = (0;1;0),\ \underline{e}^3 = (0;0;1).$$



Clearly, if the original d-dimensional walk returns to $\underline{0} = (0, \dots 0)$ then the projected walk returns to $(0, 0, 0)$. Hence, if the original d-dimensional walk (X_n^d) is recurrent then the projected chain $\left(X_n^{\mathrm{proj}}\right)$ is recurrent. But then consider the random walk on $\mathbb{Z}^3$ obtained from $\left(X_n^{\mathrm{proj}}\right)$ by discarding the stays and recording the jumps only. The latter is the nearest-neighbour symmetric random walk on $\mathbb{Z}^3$ which is transient. By Theorem 6.3 below, $\left(X_n^{\mathrm{proj}}\right)$ is also transient. Then so is (X_n^d).

Nearest-neighbour symmetric random walks are often called simple walks. Re-phrasing a famous saying, we could state that in two dimensions every road of a simple random walk will lead you to the origin (or any other given site) while in three dimensions and higher it is no longer so. The difference between two and three dimensions emerges in a countless variety of situations in virtually all domains of mathematics.

Example 5.4 (Math Tripos, Markov Chains, Part IIA, 2003, A101J and Part IIB, 2003, B101J) **(i)** Let (X_n, Y_n) be a simple symmetric random walk in $\mathbb{Z}^2$, starting from $(0, 0)$, and set $T = \inf\{n \geq 0 : \max\{|X_n|, |Y_n|\} = 2\}$. Determine the quantities $\mathbb{E}T$ and $\mathbb{P}(X_T = 2 \text{ and } Y_T = 0)$.

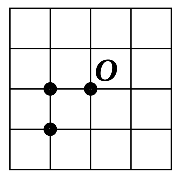
(ii) Let $(X_n)_{n\geq 0}$ be a Markov chain with state space I and transition matrix P. What does it mean to say that a state $i \in I$ is recurrent? Prove that i is recurrent if and only if $\sum\limits_{n=0}^{\infty} p_{ii}^{(n)} = \infty$, where $p_{ii}^{(n)}$ denotes the (i, i) entry in P^n.

Show that the simple symmetric random walk in $\mathbb{Z}^2$ is recurrent.

Solution **(i)** If $k_i = \mathbb{E}_i T$ and $h_i = \mathbb{P}_i(X_T Y_T = 0)$ then

$$\begin{aligned}
&k_{(0,0)} = 1 + k_{(-1,0)},\\
&k_{(-1,0)} = 1 + k_{(0,0)}/4 + k_{(-1,-1)}/2,\ k_{(-1,-1)} = 1 + k_{(-1,0)}/2,\\
&h_{(0,0)} = h_{(-1,0)},\\
&h_{(-1,0)} = 1/4 + h_{(0,0)}/4 + h_{(-1,-1)}/2,\ h_{(-1,-1)} = h_{(-1,0)}/2,
\end{aligned}$$

by conditioning on the first step, the Markov property and symmetry.



Hence,

$$\mathbb{E}T = k_{(0,0)} = \frac{9}{2},\ h_{(0,0)} = \frac{1}{2}.$$

By symmetry,

$$\mathbb{P}(X_T = 2 \text{ and } Y_T = 0) = \frac{1}{4}\ h_{(0,0)} = \frac{1}{8}.$$

(ii) State i is recurrent if $f_i = \mathbb{P}_i(T_i < \infty) = 1$ where $T_i = \inf\{n \geq 1 : X_n = i\}$. If V_i is the total time spent in i then

$$\begin{aligned}
\mathbb{P}_i(V_i \geq k+1) &= \mathbb{P}_i(V_i \geq k)\ \mathbb{P}_i(V_i \geq k+1 | V_i \geq k)\\
&= \mathbb{P}_i(V_i \geq k) f_i = \ldots = f_i^{k+1}.
\end{aligned}$$

Then

$$\mathbb{E}_i(V_i) = \sum_{k\geq 1} \mathbb{P}(V_i \geq k) = \sum_{k\geq 0} f_i^k.$$

On the other hand,

$$\mathbb{E}_i V_i = \mathbb{E}_i \sum_{n\geq 0} \mathbf{1}(X_n = i) = \sum_{n\geq 0} p_{ii}^{(n)}.$$

Hence, $f_i = 1$ if and only if $\sum_{n\geq 0} p_{ii}^{(n)} = \infty$.

Now let (X_n) be a simple symmetric random walk in $\mathbb{Z}^2$. It is irreducible, hence it suffices to check that $\sum_{n\geq 0} p_{ii}^{(n)} = \infty$ for a single $i \in \mathbb{Z}^2$, say the origin $(0,0)$. Write $(X_n^{\pm})$ for the projection of (X_n) on the diagonal $\{x = \pm y\}$ in $\mathbb{Z}^2$. Then $(X_n^{\pm})$ are independent simple symmetric random walks on $\frac{1}{\sqrt{2}}\mathbb{Z}$, and return to $(0,0)$ in (X_n) means return to 0 in each of $(X_n^{\pm})$. Next,

$$\mathbb{P}_0(X_{2k}^{\pm} = 0) = \binom{2k}{k}\frac{1}{2^{2k}},$$

and

$$p_{00}^{(2k)} = \mathbb{P}_0(X_{2k}^{+} = 0)\mathbb{P}_0(X_{2k}^{-} = 0).$$

Then Stirling's formula asserts that $p_{00}^{(2k)}$ is

$$\approx \left(\frac{\sqrt{2}}{\sqrt{2\pi k}}\frac{(2k)^{2k}}{k^{2k}}\frac{1}{2^{2k}}\right)^2 = \frac{1}{\pi k}, \quad \text{as } k \to \infty.$$

Hence,

$$\sum_{n\geq 0} p_{00}^{(n)} = \sum_{k\geq 0} p_{00}^{(2k)} = \infty.$$

Example 5.5 (Math Tripos, Markov Chains, Part IIA, 1994, A101K)
(i) Let $(X_n)_{n\geq 0}$ be a simple symmetric random walk on the integers. Show that $(X_n)_{n\geq 0}$ is recurrent.

(ii) Let $(X_n)_{n\geq 0}$, $(Y_n)_{n\geq 0}$ and $(Z_n)_{n\geq 0}$ be simple symmetric random walks on the integers. Then $V_n = (X_n, Y_n, Z_n)$ is a Markov chain. What are the transition probabilities for this chain?

Show that, with probability one, $(V_n)_{n\geq 0}$ visits $(0,0,0)$ only finitely many times.

[*Stirling's formula states*

$$n!(e/n)^n/\sqrt{2\pi n} \to 1 \ \ as \ \ n \to \infty.$$

Other standard results may be used without proofs if clearly stated.]

Solution. **(i)** We use the following result: $\forall$ state i:

$$\text{if } \mathbf{P}_i(T_i < \infty) = 1, \text{ then } \sum_{n=0}^{\infty} p_{ii}^{(n)} = \infty, \text{ and } i \text{ is R},$$

$$\text{if } \mathbf{P}_i(T_i < \infty) < 1, \text{ then } \sum_{n=0}^{\infty} p_{ii}^{(n)} < \infty, \text{ and } i \text{ is T}.$$

Also, recurrence and transience are class properties, and the simple symmetric random walk defines an irreducible chain. So, it suffices to check recurrence or transience for a single state, say 0. We have:

$$\sum_n p_{00}^{(n)} = \sum_n p_{00}^{(2n)} = \sum_n \frac{(2n)!}{n!n!} \left(\frac{1}{2}\right)^{2n}$$

and use Stirling's formula to obtain the summands

$$\frac{\sqrt{4\pi n}}{2\pi n} \frac{(2n/e)^{2n}}{(n/e)^{2n}} \frac{1}{2^{2n}} = \frac{1}{\sqrt{\pi n}}.$$

As the series $\sum_n (1/\sqrt{\pi n})$ diverges, the random walk is R.

(ii) The transition probabilities for (V_n) are

$$p_{(i,j,k)(l,m,n)} = \begin{matrix} 1/8, & \text{if } |i-l| = |j-m| = |l-n| = 1, \\ 0, & \text{otherwise.} \end{matrix} ,$$

and we aim to show that

$$\sum_{n=0}^{\infty} p_{(0,0,0)(0,0,0)}^{(n)} < \infty.$$

As before, $p_{(0,0,0)(0,0,0)}^{(n)} = 0$ when n is odd. Furthermore, $p_{(0,0,0)(0,0,0)}^{(2n)} = \left(p_{00}^{(2n)}\right)^3$ where $p_{00}^{(2n)}$ is as in part **(i)**. Hence, $\left(p_{00}^{(2n)}\right)^3 \sim \frac{1}{(\pi n)^{3/2}}$, and the series converges. So, (V_n) is T; hence the answer.

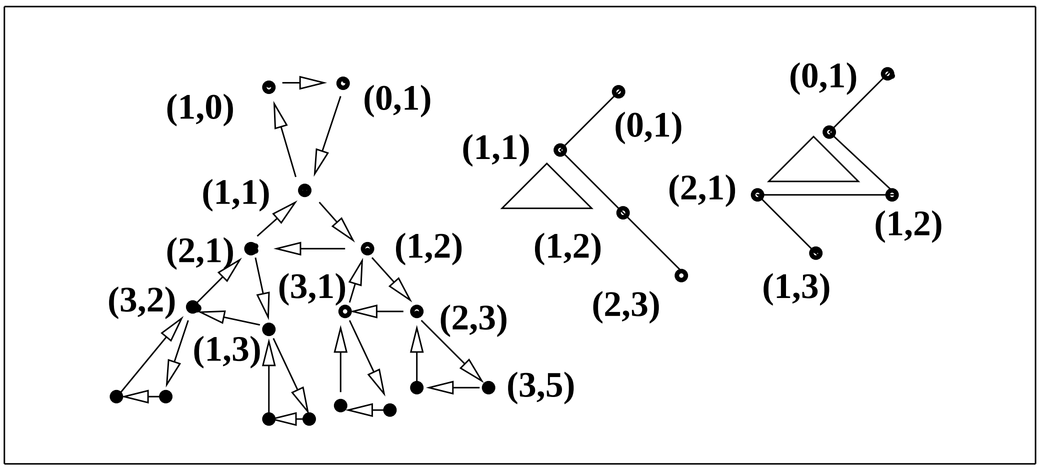
Example 5.6 (Math Tripos, Markov Chains, Part IIA, 1996, A301E) **(i)** A random sequence of non-negative integers $(F_n)_{n\geq 0}$ is obtained by setting

$F_0 = 0$ and $F_1 = 1$ and, once $F_0, \dots, F_n$ are known, taking F_{n+1} to be either the sum or the difference of F_{n-1} and F_n, each with probability $1/2$. Is $(F_n)_{n \geq 0}$ a Markov chain?

By considering the Markov chain $X_n = (F_{n-1}, F_n)$, find the probability that $(F_n)_{n \geq 0}$ reaches 3 before first returning to 0.

(ii) Draw enough of the flow diagram for $(X_n)_{n \geq 0}$ to establish a general pattern. Hence, using the strong Markov property, show that the hitting probability for $(1, 1)$, starting from $(1, 2)$, is $(3 - \sqrt{5})/2$.

Solution. See the diagram.



(F_n) is not a Markov chain, as F_{n+1} depends on F_n and F_{n-1}, but the pair (F_{n-1}, F_n) is. The initial part of the diagram shows that the level $F_n = 3$ can be reached from $(F_0, F_1) = (0, 1)$ either at $(2, 3)$ or $(1, 3)$. To hit this level before visiting level $F_n = 0$ (i.e., $(1, 0)$), we have two straight paths, supplemented with a number of adjacent triangular cycles. The first possibility gives the probability

$$1 \cdot \frac{1}{2} \cdot \frac{1}{2} \left(1 + \frac{1}{8} + \frac{1}{8^2} + \dots \right) = \frac{2}{7},$$

and the second

$$1 \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \left(1 + \frac{1}{8} + \frac{1}{8^2} + \dots \right) = \frac{1}{7},$$

which adds to $3/7$.

One can see a triangular 'pattern' emerging from the diagram, with tree-like symmetries. In particular,

$$\mathbb{P}_{(1,2)}\big(\text{hit } \ (1,1)\big) = \mathbb{P}_{(2,3)}\big(\text{hit } \ (1,2)\big) = \mathbb{P}_{(1,3)}\big(\text{hit } \ (2,1)\big) := p$$

and

$$\mathbb{P}_{(2,1)}\big(\text{hit } \ (1,1)\big) = \mathbb{P}_{(3,2)}\big(\text{hit } \ (2,1)\big) := p'.$$

Conditioning on the first jump, by the strong Markov property, we can write

$$p = \frac{1}{2}p' + \frac{1}{2}\mathbb{P}_{(2,3)}\big(\text{hit } \ (1,1)\big) = \frac{1}{2}p' + \frac{1}{2}p^2,$$

and

$$p' = \frac{1}{2} + \frac{1}{2}\mathbb{P}_{(1,3)}\big(\text{hit } \ (1,1)\big) = \frac{1}{2} + \frac{1}{2}pp' \ \ \textit{whence} \ \ p' = \frac{1}{2-p}.$$

This yields

$$p = \frac{1}{2(2-p)} + \frac{1}{2}p^2, \ \ \text{i.e.,} \ \ p^3 - 4p^2 + 4p - 1 = (p-1)(p^2 - 3p + 1) = 0.$$

The roots are $p = 1$ and $(3 \pm \sqrt{5})/2$. We are interested in the minimal non-negative root, i.e., $p = (3 - \sqrt{5})/2$.

Consequently, $p' = 2/(1 + \sqrt{5})$. Obviously, $0 < p, p' < 1$.