

MARKOV CHAINS

3. Markov chain equations

Calculation of hitting probabilities and mean hitting times; survival probability for birth and death chains.

Recall, from now on \mathbb{P}_i stands for the probability distribution generated by the (δ_i, P) Markov chain, starting from state $i \in I$, and \mathbb{E}_i for the corresponding expectation.

Let $A \subset I$ be a set of states. The *hitting time* H^A is the first time the Markov chain enters A :

$$H^A = \inf \{n \geq 0 : X_n \in A\}. \quad (3.1)$$

The hitting probability h_i^A is the probability that the chain starting from state i will ever hit A :

$$h_i^A = \mathbb{P}_i(H^A < \infty); \quad (3.2)$$

when A is a closed class, h_i^A is called the absorption probability. The expected value of H^A is denoted by k_i^A :

$$k_i^A = \mathbb{E}_i(H^A) = \sum_{0 < n < \infty} n \mathbb{P}_i(H^A = n) + \infty \cdot \mathbb{P}_i(H^A = \infty) \quad (3.3)$$

so that if $\mathbb{P}_i(H^A = \infty) > 0$, $k_i^A = \infty$.

The basis for calculating the hitting probabilities is provided by

Theorem 3.1. *Given $A \subset I$, the h_i^A 's give the minimal non-negative solutions to the following linear system:*

$$\begin{cases} h_i^A \equiv 1, & i \in A, \\ h_i^A = \sum_{j \in I} p_{ij} h_j^A, & i \notin A. \end{cases} \quad (3.4)$$

That is, if $g_i \geq 0$ is any solution then $g_i \geq h_i^A$, $i \in I$.

Proof. Recall, h_i^A is calculated for $X_0 = i$. When $i \in A$, $H^A = 0$, so $h_i^A = 1$. If $i \notin A$, $H^A \geq 1$, and

$$\begin{aligned} h_i^A &= \sum_{j \in I} \mathbb{P}_i(H^A < \infty, X_1 = j) = \sum_j \mathbb{P}_i(X_1 = j) \mathbb{P}_i(H^A < \infty | X_1 = j) \\ &= (\text{by the Markov property}) \sum_j p_{ij} \mathbb{P}_j(H^A < \infty) = \sum_j p_{ij} h_j^A. \end{aligned} \quad (3.5)$$

Now take any non-negative solution g_i . For $i \in A$, $g_i = h_i^A = 1$. For $i \notin A$:

$$\begin{aligned} g_i &= \sum_j p_{ij} g_j = \sum_{j \in A} p_{ij} + \sum_{j \notin A} p_{ij} g_j \\ &= \sum_{i \in A} p_{ij} + \sum_{j \notin A} p_{ij} \left(\sum_{k \in A} p_{jk} + \sum_{k \notin A} p_{jk} g_k \right) \\ &= \mathbb{P}_i(X_1 \in A) + \mathbb{P}_i(X_1 \notin A, X_2 \in A) + \sum_{j \in A, k \notin A} p_{ij} p_{jk} g_k. \end{aligned}$$

By repeated substitution, $\forall n$:

$$\begin{aligned} g_i &= \mathbb{P}_i(X_1 \in A) + \dots + \mathbb{P}_i(X_1 \notin A, \dots, X_{n-1} \notin A, X_n \in A) \\ &\quad + \sum_{j_1 \notin A} \dots \sum_{j_n \notin A} p_{ij_1} p_{j_1 j_2} \dots p_{j_{n-1} j_n} g_{j_n}. \end{aligned} \quad (3.6)$$

As $g_i \geq 0$, omitting the last sum makes the right-hand side smaller. The first n summands give $\mathbb{P}_i(H^A \leq n)$. Hence:

$$g_i \geq \mathbb{P}_i(H^A \leq n), \quad \forall n \geq 0. \quad (3.7)$$

Then:

$$g_i \geq \lim_{n \rightarrow \infty} \mathbb{P}_i(H^A \leq n) = \mathbb{P}_i(H^A < \infty) = h_i^A. \quad (3.8)$$

□

In general, you can write equations for more intricate hitting probabilities. They often give us a powerful tool, especially when a symmetry of a Markov chain can be used. See Examples below.

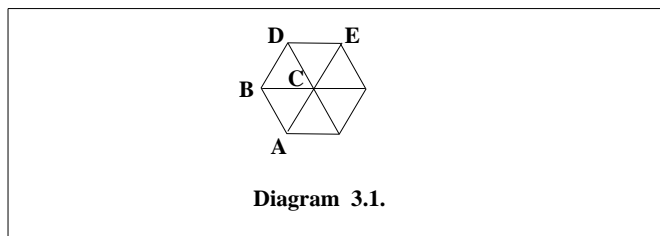
Examples and remarks. 3.1. $h_i \equiv 1$ is always a (strictly positive) solution to (3.4). However, it is not always minimal among non-negative solutions (although sometimes it is).

3.2. (Math Tripos, Part IB, 1993, Question 502K; the initial part) Construct a graph on seven vertices as follows: take a regular hexagon and join opposite corners by a straight line; let the vertices be the corners of the hexagon together with the point at the centre; let the edges be the perimeter of the hexagon together with the lines joining the corners to the centre. At discrete intervals a particle moves from one vertex of this graph to one of the adjacent vertices at random, and independently of past moves. Suppose the particle starts at a corner A. Find the probability that the particle will return to A without hitting the central vertex C.

Solution. See Diagram 3.1. Set:

$$h_i = \mathbb{P}_i(\text{hit A before C}).$$

Then the probability in question is h_A , and



by the symmetry:

$$h_A = \frac{2}{3}h_B.$$

Now: (a)

$$h_B = \frac{1}{3} + \frac{1}{3}h_D, \quad h_D = \frac{1}{3}h_B + \frac{1}{3}h_E,$$

and again by the symmetry,

$$h_E = \frac{2}{3}h_D.$$

Then

$$h_D = \frac{1}{3}h_B + \frac{2}{9}h_D, \quad \text{i.e., } h_D = \frac{3}{7}h_B.$$

Next,

$$h_B = \frac{1}{3} + \frac{1}{7}h_B, \quad \text{i.e., } h_B = \frac{7}{18}.$$

Hence, $h_A = 7/27$.

3.3. (A homogeneous birth and death process; see Example 2.4 b). First, consider the homogeneous birth and death process described in Example 2.4 b). Set $h_i = \mathbb{P}_i(\text{hit 0})$ (the extinction probability, starting from state i), then h_i is the minimal non-negative solution to

$$h_0 = 1, \quad h_i = ph_{i+1} + (1-p)h_{i-1}, \quad i \geq 1. \quad (3.9)$$

For $p \neq 1/2$, this is solved by

$$h_i = A + B \left(\frac{1-p}{p} \right)^i. \quad (3.10)$$

If $p < 1/2$ then minimality and non-negativity imply that $B = 0$ and $A = 1$, with $h \equiv 1$. If $p > 1/2$, the conclusion is that $A = 0$ and $B = 1$, with

$$h_i = \left(\frac{1-p}{p} \right)^i.$$

For $p = 1/2$, the solution has the form

$$h_i = A + Bi$$

and again the minimality and non-negativity imply that $B = 0$ and $A = 1$, with $h_i \equiv 1$.

Note that the answers do not depend on the choice of p_{0j} , transition probabilities from state 0. Also, h_i is the extinction and $1 - h_i$ the survival probability (conditional on $X_0 = i$). Therefore, the survival probabilities are

$$1 - h_i \begin{cases} \equiv 0, & \text{for } 0 \leq p \leq 1/2, \\ = 1 - \left(\frac{1-p}{p} \right)^i, \quad i \geq 0, & \text{for } 1/2 < p \leq 1. \end{cases} \quad (3.11)$$

3.4. Next, we move to an inhomogeneous birth and death process from Example 2.4 c). Here the equations become state-dependent:

$$h_0 = 1, \quad h_i = p_i h_{i+1} + (1 - p_i) h_{i-1}, \quad i \geq 1. \quad (3.12)$$

The way to solve it is to pass to the differences

$$u_i = h_{i-1} - h_i, \text{ with } p_i u_{i+1} = (1 - p_i) u_i,$$

and

$$u_{i+1} = \frac{1 - p_i}{p_i} u_i = \frac{1 - p_i}{p_i} \frac{1 - p_{i-1}}{p_{i-1}} \dots \frac{1 - p_1}{p_1} u_1.$$

Set: $\gamma_i = \frac{1 - p_{i-1}}{p_{i-1}} \dots \frac{1 - p_1}{p_1}$; then as

$$u_1 + \dots + u_i = h_0 - h_i,$$

we obtain:

$$h_i = 1 - A(\gamma_0 + \dots + \gamma_{i-1}). \quad (3.13)$$

Here, $\gamma_0 = 1$ and $A = u_1$. Constant A has to be determined from the condition of non-negative minimality:

$$A = \left(\sum_{i=0}^{\infty} \gamma_i \right)^{-1}.$$

That is,

$$h_i \equiv 1, \text{ if } \sum_{j=0}^{\infty} \gamma_j = \infty,$$

and

$$h_i = \sum_{j=i}^{\infty} \gamma_j / \sum_{j=0}^{\infty} \gamma_j, \text{ if } \sum_{j=0}^{\infty} \gamma_j < \infty.$$

In particular, in the second case, $h_{i+1} \leq h_i$ and $\lim_{i \rightarrow \infty} h_i = 0$. The survival probabilities become

$$\begin{cases} 1 - \sum_{j=i}^{\infty} \gamma_j / \sum_{j=0}^{\infty} \gamma_j, & \text{if } \sum_{j=0}^{\infty} \gamma_j < \infty, \\ 0, & \text{if } \sum_{j=0}^{\infty} \gamma_j = \infty. \end{cases} \quad (3.14)$$

Passing to the mean hitting times k_i^A , we have

Theorem 3.2. Given $A \subset I$, the k_i^A 's give the minimal non-negative solutions to the following linear system:

$$\begin{cases} k_i^A \equiv 0, & i \in A, \\ k_i^A = 1 + \sum_{j \notin A} p_{ij} k_j^A, & i \notin A. \end{cases} \quad (3.15)$$

That is, if $g_i \geq 0$ is any solution then $g_i \geq k_i^A$, $i \in I$.

Proof. Like h_i^A before, the expected hitting time k_i^A is calculated for $X_0 = i$. When $i \in A$, $h^A = 0$, so $k_i^A = 0$. If $i \notin A$, $H^A \geq 1$, and

$$\mathbb{E}_i(H^A | X_1 = j) = 1 + \mathbb{E}_j H^A$$

by the Markov property. Thus,

$$\begin{aligned} k_i^A &= \mathbb{E}_i(H^A) = \sum_{j \in I} \mathbb{E}_i(H^A \mathbf{1}(X_1 = j)) = \sum_j \mathbb{P}_i(X_1 = j) \mathbb{E}_i(H^A | X_1 = j) \\ &= 1 + \sum_{j \notin A} \mathbb{P}_i(X_1 = j) \mathbb{E}_i(H^A | X_1 = j) = 1 + \sum_{j \notin A} p_{ij} k_j^A. \end{aligned}$$

Let now g_i be any non-negative solution. Then $g_i = k_i^A = 0$ for $i \in A$. If $i \notin A$:

$$g_i = 1 + \sum_{j \notin A} p_{ij} g_j = 1 + \sum_{j \notin A} p_{ij} \left(1 + \sum_{k \notin A} p_{jk} g_k \right).$$

Writing 1 as $\mathbb{P}_i(H^A \geq 1)$ and $\sum_{j \notin A} p_{ij}$ as $\mathbb{P}(H^A \geq 2)$, obtain:

$$g_i = \mathbb{P}_i(H^A \geq 1) + \mathbb{P}_i(H^A \geq 2) + \sum_{j \notin A} p_{ij} + \sum_{k \notin A} p_{jk} g_k.$$

By repeated substitution, $\forall n$:

$$\begin{aligned} g_i &= 1 + \mathbb{P}_i(H^A \geq 1) + \dots + \mathbb{P}_i(H^A \geq n) + \sum_{j_1 \notin A} \dots \sum_{j_n \notin A} p_{ij_1} p_{j_1 j_2} \dots p_{j_{n-1} j_n} g_{j_n} \\ &\geq \mathbb{P}_i(H^A \geq 1) + \dots + \mathbb{P}_i(H^A \geq n) \end{aligned} \quad (3.16)$$

as $g_i \geq 0$. Then, as $n \rightarrow \infty$,

$$g_i \geq \sum_{n=1}^{\infty} \mathbb{P}_i(H^A \geq n) = \mathbb{E}_i H^A = k_i^A. \quad (3.17)$$

□

As in the case of the h_i^A 's, equations (3.15) can be very efficient, especially when the system has symmetries.

Examples and remarks. 3.5. Note that the only non-negative solution to (3.15) may be $k_i^A \begin{cases} \equiv 0, & i \in A, \\ \equiv +\infty, & i \notin A. \end{cases}$ See Examples 3.6 and 3.7 below.

3.6. For the homogeneous birth and death process from Example 2.4 b), set: $k_i = \mathbb{E}_i(H^{(0)})$, the expected time of hitting 0. Then k_i is the minimal non-negative solution to

$$k_0 = 0, \quad k_i = 1 + pk_{i+1} + (1-p)k_{i-1}, \quad i \geq 1.$$

The general solution here is of the form $k_i = A + Bi$; the constants A and B are given by $A = 0$, $B = 1/(1-2p)$. However, for $p \geq 1/2$, there is no finite non-negative solution. Hence, for $i \geq 1$:

$$k_i \begin{cases} = \left(\frac{1}{1-2p}\right) i, & \text{for } 0 \leq p < 1/2, \\ \equiv +\infty, & \text{for } 1/2 \leq p \leq 1. \end{cases} \quad (3.18)$$

Note a fine phenomenon specified in (3.11) and (3.18) for $p = 1/2$: according to (3.11), $p = 1/2$ has the extinction probability $h_i \equiv 1$ but, in view of (3.18), the mean extinction time $k_i \equiv +\infty$ (even when you start at $i = 1$, the nearest neighbour of 0). This yields an important example of a random variables $H^{(0)}$ (time of hitting state 0) with $\mathbb{P}_i(H^{(0)} < +\infty) \equiv 1$ (i.e., $H^{(0)}$ is finite with \mathbb{P}_i -probability one) but $\mathbb{E}_i H^{(0)} \equiv +\infty$.

3.7. (Math Tripos, Part IIA, 1995, Question A101M; part (ii)) A flight of stairs has N steps. A frog starts at the bottom of the stairs and tries to jump to the top, making a series of independent jumps as follows. When the frog is on the i^{th} step ($0 < i < N$) it succeeds in jumping up to step $i + 1$ with probability α ($0 < \alpha < 1/2$), but with probability α it falls down to step $i - 1$ and with probability $1 - 2\alpha$ it lands again on the i^{th} step. When the frog is at the bottom of the stairs (on step 0) it succeeds in jumping up to step 1 with probability β ($0 < \beta < 1$), but with probability $1 - \beta$ it remains where it is. What is the expected number of jumps before the frog reaches the top of the stairs?

Suppose that the same frog starts N steps below the top of an infinite flight of stairs. What now is the expected number of jumps before the frog reaches the top of the stairs?

Solution. Set

$$k_i = \mathbb{E}(\text{time to reach the top} \mid \text{starts } i \text{ steps below}), \quad i \geq 0.$$

The system of equations for the $\overline{0, N}$ flight is

$$\begin{aligned} k_0 &= 0, \\ k_i &= 1 + \alpha k_{i-1} + (1 - 2\alpha)k_i + \alpha k_{i+1}, \quad 1 \leq i \leq N - 1, \\ k_N &= 1 + (1 - \beta)k_N + \beta k_{N-1}. \end{aligned}$$

Here, the general solution is

$$k_i = A + Bi - \frac{1}{2\alpha}i^2,$$

and the boundary equations at $i = 0$ and $i = N$ yield

$$k_i = \frac{N^2 - (N - i)^2}{2\alpha} - \frac{i}{2\alpha} + \frac{N}{\beta}, \quad 1 \leq i \leq N,$$

with

$$k_N = \frac{N(N - 1)}{2\alpha} + \frac{N}{\beta}.$$

For infinite stairs, $A + Bi - \frac{1}{2\alpha}i^2$ cannot be maintained non-negative. Hence, $k_i \equiv \infty$ for $i \geq 1$.