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MARKOV CHAINS

3. Markov chain equations

Calculation of hitting probabilities and mean hitting times; survival probability for birth and death chains.

Recall, from now on $\mathbb{P}_i$ stands for the probability distribution generated by the (δ_i, P) Markov chain, starting from state $i \in I$, and $\mathbb{E}_i$ for the corresponding expectation.

Let $A \subset I$ be a set of states. The *hitting* time H^A is the first time the Markov chain enters A:

$$H^A = \inf\,\{n \geq 0 :\ X_n \in A\}. \tag{3.1}$$

The hitting probability h_i^A is the probability that the chain starting from state i will ever hit A:

$$h_i^A = \mathbb{P}_i(H^A < \infty); \tag{3.2}$$

when A is a closed class, h_i^A is called the absorption probability. The expected value of H^A is denoted by k_i^A:

$$k_i^A = \mathbb{E}_i(H^A) = \sum_{0<n<\infty} n\mathbb{P}_i(H^A = n) + \infty \cdot \mathbb{P}_i(H^A = \infty) \tag{3.3}$$

so that if $\mathbb{P}_i(H^A = \infty) > 0$, $k_i^A = \infty$.

The basis for calculating the hitting probabilities is provided by

Theorem 3.1. *Given $A \subset I$, the h_i^A's give the minimal non-negative solutions to the following linear system:*

$$\begin{cases} h_i^A \equiv 1, & i \in A, \\ h_i^A = \sum\limits_{j \in I} p_{ij} h_j^A, & i \notin A. \end{cases} \tag{3.4}$$

That is, if $g_i \geq 0$ is any solution then $g_i \geq h_i^A$, $i \in I$.

Proof. Recall, h_i^A is calculated for $X_0 = i$. When $i \in A$, $H^A = 0$, so $h_i^A = 1$. If $i \notin A$, $H^A \geq 1$, and

$$\begin{aligned} h_i^A &= \sum_{j \in I} \mathbb{P}_i(H^A < \infty, X_1 = j) = \sum_j \mathbb{P}_i(X_1 = j)\mathbb{P}_i(H^A < \infty | X_1 = j) \\ &= \text{(by the Markov property) } \sum_j p_{ij}\mathbb{P}_j(H^A < \infty) = \sum_j p_{ij} h_j^A. \end{aligned} \tag{3.5}$$

Now take any non-negative solution g_i. For $i \in A$, $g_i = h_i^A = 1$. For $i \notin A$:

$$\begin{aligned} g_i &= \sum_j p_{ij} g_j = \sum_{j \in A} p_{ij} + \sum_{j \notin A} p_{ij} g_j \\ &= \sum_{i \in A} p_{ij} + \sum_{j \notin A} p_{ij} \left(\sum_{k \in A} p_{jk} + \sum_{k \notin A} p_{jk} g_k \right) \\ &= \mathbb{P}_i(X_1 \in A) + \mathbb{P}_i(X_1 \notin A, X_2 \in A) + \sum_{j \in A, k \notin A} p_{ij} p_{jk} g_k. \end{aligned}$$

By repeated substitution, $\forall\, n$:

$$\begin{aligned} g_i &= \mathbb{P}_i(X_1 \in A) + \ldots + \mathbb{P}_i(X_1 \notin A, \ldots, X_{n-1} \notin A, X_n \in A) \\ &\qquad\qquad + \sum_{j_1 \notin A} \cdots \sum_{j_n \notin A} p_{ij_1} p_{j_1 j_2} \cdots p_{j_{n-1} j_n} g_{j_n}. \end{aligned} \tag{3.6}$$

As $g_i \geq 0$, omitting the last sum makes the right-hand side smaller. The first n summands give $\mathbb{P}_i(H^A \leq n)$. Hence:

$$g_i \geq \mathbb{P}_i(H^A \leq n), \quad \forall\, n \geq 0. \tag{3.7}$$

Then:

$$g_i \geq \lim_{n \to \infty} \mathbb{P}_i(H^A \leq n) = \mathbb{P}_i(H^A < \infty) = h_i^A. \tag{3.8}$$

□

In general, you can write equations for more intricate hitting probabilities. They often give us a powerful tool, especially when a symmetry of a Markov chain can be used. See Examples below.

Examples and remarks. 3.1. $h_i \equiv 1$ is always a (strictly postitive) solution to (3.4). However, it is not always minimal among non-negative solutions (although sometimes it is).

3.2. (Math Tripos, Part IB, 1993, Question 502K; the initial part) Construct a graph on seven vertices as follows: take a regular hexagon and join opposite corners by a straight line; let the vertices be the corners of the hexagon together with the point at the centre; let the edges be the perimeter of the hexagon together with the lines joining the corners to the centre. At discrete intervals a particle moves from one vertex of this graph to one of the adjacent vertices at random, and independently of past moves. Suppose the particle starts at a corner A. Find the probability that the particle will return to A without hitting the central vertex C.

Solution. See Diagram 3.1. Set:

$$h_i = \mathbb{P}_i(\text{hit A before C}).$$

Then the probability in question is h_{A}, and

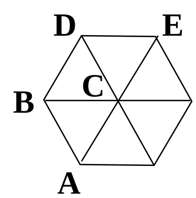

Diagram 3.1.

by the symmetry:

$$h_{\text{A}} = \frac{2}{3} h_{\text{B}}.$$

Now:*(a)*

$$h_{\text{B}} = \frac{1}{3} + \frac{1}{3} h_{\text{D}}, \quad h_{\text{D}} = \frac{1}{3} h_{\text{B}} + \frac{1}{3} h_{\text{E}},$$

and again by the symmetry,

$$h_{\text{E}} = \frac{2}{3} h_{\text{D}}.$$

Then

$$h_{\text{D}} = \frac{1}{3} h_{\text{B}} + \frac{2}{9} h_{\text{D}}, \quad \text{i.e.,} \quad h_{\text{D}} = \frac{3}{7} h_{\text{B}}.$$

Next,

$$h_{\mathrm{B}} = \frac{1}{3} + \frac{1}{7}h_{\mathrm{B}}, \ \text{ i.e., } \ h_{\mathrm{B}} = \frac{7}{18}.$$

Hence, $h_{\mathrm{A}} = 7/27$.

3.3. (A homogeneous birth and death process; see Example 2.4 b). First, consider the homogeneous birth and death process described in Example 2.4 b). Set $h_i = \mathbb{P}_i(\text{hit } 0)$ (the extinction probability, starting from state i), then h_i is the minimal non-negative solution to

$$h_0 = 1, \ \ h_i = p h_{i+1} + (1-p) h_{i-1}, \ i \geq 1. \tag{3.9}$$

For $p \neq 1/2$, this is solved by

$$h_i = A + B\left(\frac{1-p}{p}\right)^i. \tag{3.10}$$

If $p < 1/2$ then minimality and non-negativity imply that $B = 0$ and $A = 1$, with $h \equiv 1$. If $p > 1/2$, the conclusion is that $A = 0$ and $B = 1$, with

$$h_i = \left(\frac{1-p}{p}\right)^i.$$

For $p = 1/2$, the solution has the form

$$h_i = A + Bi$$

and again the minimality and non-negativity imply that $B = 0$ and $A = 1$, with $h_i \equiv 1$.

Note that the answers do not depend on the choice of p_{0j}, transistion probabilities from state 0. Also, h_i is the extinction and $1 - h_i$ the survival probability (conditional on $X_0 = i$). Therefore, the survival probabilities are

$$1 - h_i \begin{cases} \equiv 0, & \text{for} \quad 0 \leq p \leq 1/2, \\ = 1 - \left(\frac{1-p}{p}\right)^i, \ i \geq 0, & \text{for} \quad 1/2 < p \leq 1. \end{cases} \tag{3.11}$$

3.4. Next, we move to an inhomogeneous birth and death process from Example 2.4 c). Here the equations become state-dependent:

$$h_0 = 1, \ \ h_i = p_i h_{i+1} + (1-p_i) h_{i-1}, \ i \geq 1. \tag{3.12}$$

The way to solve it is to pass to the differences

$$u_i = h_{i-1} - h_i, \quad \text{with} \;\; p_i u_{i+1} = (1-p_i)u_i,$$

and

$$u_{i+1} = \frac{1-p_i}{p_i} u_i = \frac{1-p_i}{p_i}\frac{1-p_{i-1}}{p_{i-1}} \cdots \frac{1-p_1}{p_1} u_1.$$

Set: $\gamma_i = \dfrac{1-p_{i-1}}{p_{i-1}} \cdots \dfrac{1-p_1}{p_1}$; then as

$$u_1 + \ldots + u_i = h_0 - h_i,$$

we obtain:

$$h_i = 1 - A\big(\gamma_0 + \ldots + \gamma_{i-1}\big). \tag{3.13}$$

Here, $\gamma_0 = 1$ and $A = u_1$. Constant A has to be determined from the condition of non-negative minimality:

$$A = \left(\sum_{i=0}^{\infty} \gamma_i\right)^{-1}.$$

That is,

$$h_i \equiv 1, \;\; \text{if} \;\; \sum_{j=0}^{\infty} \gamma_j = \infty,$$

and

$$h_i = \sum_{j=i}^{\infty} \gamma_j \Bigg/ \sum_{j=0}^{\infty} \gamma_j, \;\; \text{if} \;\; \sum_{j=0}^{\infty} \gamma_j < \infty.$$

In particular, in the second case, $h_{i+1} \leq h_i$ and $\lim_{i\to\infty} h_i = 0$. The survival probabilities become

$$\begin{cases} 1 - \sum\limits_{j=i}^{\infty} \gamma_j \Big/ \sum\limits_{j=0}^{\infty} \gamma_j, & \text{if} \;\; \sum\limits_{j=0}^{\infty} \gamma_j < \infty, \\ 0, & \text{if} \;\; \sum\limits_{j=0}^{\infty} \gamma_j = \infty. \end{cases} \tag{3.14}$$

Passing to the mean hitting times k_i^A, we have

Theorem 3.2. *Given $A \subset I$, the k_i^A's give the minimal non-negative solutions to the following linear system:*

$$\begin{cases} k_i^A \equiv 0, & i \in A, \\ k_i^A = 1 + \sum\limits_{j \notin A} p_{ij} k_j^A, & i \notin A. \end{cases} \tag{3.15}$$

That is, if $g_i \geq 0$ is any solution then $g_i \geq k_i^A$, $i \in I$.

Proof. Like h_i^A before, the expected hitting time k_i^A is calculated for $X_0 = i$. When $i \in A$, $h^A = 0$, so $k_i^A = 0$. If $i \notin A$, $H^A \geq 1$, and

$$\mathbb{E}_i(H^A | X_1 = j) = 1 + \mathbb{E}_j H^A$$

by the Markov property. Thus,

$$k_i^A = \mathbb{E}_i(H^A) = \sum_{j \in I} \mathbb{E}_i(H^A \mathbf{1}(X_1 = j)) = \sum_j \mathbb{P}_i(X_1 = j) \mathbb{E}_i(H^A | X_1 = j)$$

$$1 + \sum_{j \notin A} \mathbb{P}_i(X_1 = j) \mathbb{E}_i(H^A | X_1 = j) = 1 + \sum_{j \notin A} p_{ij} k_j^A.$$

Let now g_i be any non-negative solution. Then $g_i = k_i^A = 0$ for $i \in A$. If $i \notin A$:

$$g_i = 1 + \sum_{j \notin A} p_{ij} g_j = 1 + \sum_{j \notin A} p_{ij} \left(1 + \sum_{k \notin A} p_{jk} g_k \right).$$

Writing 1 as $\mathbb{P}_i(H^A \geq 1)$ and $\sum\limits_{j \neq A} p_{ij}$ as $\mathbb{P}(H^A \geq 2)$, obtain:

$$g_i = \mathbb{P}_i(H^A \geq 1) + \mathbb{P}_i(H^A \geq 2) + \sum_{j \notin A} p_{ij} + \sum_{k \notin A} p_{jk} g_k.$$

By repeated substitution, $\forall\ n$:

$$\begin{aligned} g_i &= 1 + \mathbb{P}_i(H^A \geq 1) + \ldots + \mathbb{P}_i(H^A \geq n) + \sum_{j_1 \notin A} \ldots \sum_{j_n \notin A} p_{ij_1} p_{j_1 j_2} \cdots p_{j_{n-1} j_n} g_{j_n} \\ &\geq \mathbb{P}_i(H^A \geq 1) + \ldots + \mathbb{P}_i(H^A \geq n) \end{aligned} \tag{3.16}$$

as $g_i \geq 0$. Then, as $n \to \infty$,

$$g_i \geq \sum_{n=1}^{\infty} \mathbb{P}_i(H^A \geq n) = \mathbb{E}_i H^A = k_i^A. \tag{3.17}$$

□

As in the case of the h_i^A's, equations (3.15) can be very efficient, especially when the system has symmetries.

Examples and remarks. 3.5. Note that the only non-negative solution to (3.15) may be $k_i^A \begin{cases} \equiv 0, & i \in A, \\ \equiv +\infty, & i \notin A. \end{cases}$ See Examples 3.6 and 3.7 below.

3.6. For the homogeneous birth and death process from Example 2.4 b), set: $k_i = \mathbb{E}_i(H^{\{0\}})$, the expected time of hitting 0. Then k_i is the minimal non-negative solution to

$$k_0 = 0, \;\; k_i = 1 + pk_{i+1} + (1-p)k_{i-1}, \; i \geq 1.$$

The general solution here is of the form $k_i = A + Bi$; the constants A and B are given by $A = 0$, $B = 1/(1-2p)$. However, for $p \geq 1/2$, there is no finite non-negative solution. Hence, for $i \geq 1$:

$$k_i \begin{cases} = \left(\dfrac{1}{1-2p}\right) i, & \text{for} \quad 0 \leq p < 1/2, \\ \equiv +\infty, & \text{for} \quad 1/2 \leq p \leq 1. \end{cases} \tag{3.18}$$

Note a fine phenomenon specified in (3.11) and (3.18) for $p = 1/2$: according to (3.11), $p = 1/2$ has the extinction probability $h_i \equiv 1$ but, in view of (3.18), the mean extinction time $k_i \equiv +\infty$ (even when you start at $i = 1$, the nearest neighbour of 0). This yields an important example of a random variables $H^{\{0\}}$ (time of hitting state 0) with $\mathbb{P}_i(H^{\{0\}} < +\infty) \equiv 1$ (i.e., $H^{\{0\}}$ is finite with $\mathbb{P}_i$- probability one) but $\mathbb{E}_i H^{\{0\}} \equiv +\infty$.

3.7. (Math Tripos, Part IIA, 1995, Question A101M; part (ii)) A flight of stairs has N steps. A frog starts at the bottom of the stairs and tries to jump to the top, making a series of independent jumps as follows. When the frog is on the i^{th} step $(0 < i < N)$ it succeeds in jumping up to step $i+1$ with probability α $(0 < \alpha < 1/2)$, but with probability α it falls down to step $i-1$ and with probability $1-2\alpha$ it lands again on the i^{th} step. When the frog is at the bottom of the stairs (on step 0) it succeeds in jumping up to step 1 with probability β $(0 < \beta < 1)$, but with probability $1-\beta$ it remains where it is. What is the expected number of jumps before the frog reaches the top of the stairs?

Suppose that the same frog starts N steps below the top of an infinite flight of stairs. What now is the expected number of jumps before the frog reaches the top of the stairs?

Solution. Set

$$k_i = \mathbb{E}\Big(\text{time to reach the top} \,\Big|\, \text{starts } i \text{ steps below}\Big), \quad i \geq 0.$$

The system of equations for the $\overline{0, N}$ flight is

$$\begin{aligned}
&k_0 = 0,\\
&k_i = 1 + \alpha k_{i-1} + (1-2\alpha)k_i + \alpha k_{i+1}, \quad 1 \leq i \leq N-1,\\
&k_N = 1 + (1-\beta)k_N + \beta k_{N-1}.
\end{aligned}$$

Here, the general solution is

$$k_i = A + Bi - \frac{1}{2\alpha} i^2,$$

and the boundary equations at $i = 0$ and $i = N$ yield

$$k_i = \frac{N^2 - (N-i)^2}{2\alpha} - \frac{i}{2\alpha} + \frac{N}{\beta}, \quad 1 \leq i \leq N,$$

with

$$k_N = \frac{N(N-1)}{2\alpha} + \frac{N}{\beta}.$$

For infinite stairs, $A + Bi - \dfrac{1}{2\alpha} i^2$ cannot be maintained non-negative. Hence, $k_i \equiv \infty$ for $i \geq 1$.