ANDERSON LOCALIZATION FOR A MULTI-PARTICLE MODEL WITH ALLOY-TYPE EXTERNAL POTENTIAL

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Abstract. We establish exponential localization for a multi-particle Anderson model in a Euclidean space $\mathbb{R}^d$, $d \geq 1$, in presence of a non-trivial short-range interaction and an alloy-type random external potential. Specifically, we prove that all eigenfunctions with eigenvalues near the lower edge of the spectrum decay exponentially.

1. Introduction. The N-particle Hamiltonian in the continuum

1.1. **The model.** This paper considers an N-particle Anderson model in $\mathbb{R}^d$ with interaction. The Hamiltonian $\mathbf{H} = \mathbf{H}^{(N)}(\omega)$ is a random Schrödinger operator of the form

$$\mathbf{H}^{(N)}(\omega) = -\frac{1}{2}\mathbf{\Delta} + \mathbf{U} + \mathbf{V}(\omega) \tag{1.1}$$

acting on functions from $L^2(\mathbb{R}^d \times \cdots \times \mathbb{R}^d) \simeq L^2(\mathbb{R}^d)^{\otimes N}$. This means that we consider N quantum particles in $\mathbb{R}^d$. The joint position vector is $\mathbf{x} = (x_1, \ldots, x_N) \in \mathbb{R}^{Nd}$, where component $x_j = (\mathrm{x}_j^{(1)}, \ldots, \mathrm{x}_j^{(d)}) \in \mathbb{R}^d$ represents the j-th particle, $j = 1, \ldots, N$. Next,

$$-\frac{1}{2}\mathbf{\Delta} = -\frac{1}{2}\sum_{1\leq j\leq N} \mathit{\Delta}_j$$

is the standard kinetic energy operator obtained by adding up the kinetic energies $-\frac{1}{2}\mathit{\Delta}_j$ of the individual particles; here, $\mathit{\Delta}_j$ denotes the d-dimensional Laplacian.

The interaction energy operator is denoted by $\mathbf{U}$: it is the operator of multiplication by a function $\mathbb{R}^{Nd} \ni \mathbf{x} \mapsto U(\mathbf{x})$, the inter-particle potential (which can also incorporate a deterministic external potential). Finally, $\mathbf{V}(\omega)$ is the operator of multiplication by a function

$$\mathbf{x} = (x_1, \ldots, x_N) \in \mathbb{R}^{Nd} \longmapsto V(x_1; \omega) + \cdots + V(x_N; \omega), \tag{1.2}$$

where $V : \mathbb{R}^d \times \Omega \to \mathbb{R}$ is the random external field potential, relative to a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, acting on an individual particle.

Assumptions on $U(\mathbf{x})$ and $V(x;\omega)$ are discussed below, in subsections 1.3 and 1.4. In essence, U is required to be a sum of short-range inter-particle potentials while V is assumed to be of the so-called alloy type. We refer to the quantum system with Hamiltonian $\mathbf{H}$ as a multi-particle alloy-type Anderson model in $\mathbb{R}^d$.

In this paper, we analyse spectral properties of $\mathbf{H}$ by using the method called Multi-Scale Analysis (MSA), more precisely, a multi-particle adaptation of a single-particle "continuous-space" version of the MSA. Our main result is Theorem 1.5, asserting

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that with probability one the spectrum of operator $\mathbf{H}$ near its lower edge is pure point, with an exponential decay of the corresponding eigenfunctions. Such a phenomenon is known as (exponential) Anderson localisation. In the context of the alloy-type Anderson models one often refers to the famous "Lifshits-tail" picture suggesting possible localisation domains in terms of relevant parameters.

The fact that the spectrum near its lower edge is non-empty (and even dense) follows easily from the assumption that the interaction potential U has a short range, combined with known facts about spectra of single-particle Anderson-type Hamiltonians.

Theorem 1.5 is the first rigorous result on localisation in multi-particle continuous-space Anderson models.

For lattice (tight-binding) Anderson models, the multi-particle adaptation of the MSA has been developed in earlier papers [7], [8], [9]. An alternative approach based on the Fractional Moment Method (FMM) was successfully employed, for multi-particle lattice Anderson models, in [1]; see also [2].

The structure of the present paper is commented on in subsection 1.9.

1.2. **Basic notation.** Throughout this paper, we fix integers $N > 1$ and $d \geq 1$ (which can be arbitrary) and work with configurations of $n \leq N$ distinguishable quantum particles in $\mathbb{R}^d$. The configuration space of an n-particle system is the Euclidean space $\left(\mathbb{R}^d\right)^n$ which is canonically identified with $\mathbb{R}^{nd}$. A similar identification is always used for the cubic lattices: $\left(\mathbb{Z}^d\right)^n \cong \mathbb{Z}^{nd}$.

It is convenient to endow $\mathbb{R}^d$ and $\mathbb{R}^{nd}$ with max-norm:

$$|x| = \max_{1\leq i\leq d} |\mathrm{x}^{(i)}|, \quad |\mathbf{x}| = \max_{1\leq j\leq n} |x_j|. \tag{1.3}$$

The distance "dist" below is induced by this norm. In terms of the max-norm in $\mathbb{R}^d$ the ball of radius L centered at $u = (u^{(1)}, \dots, u^{(d)}$ is the the cube

$$\Lambda_L(u) := \mathop{\times}_{i=1}^{d} \left(\mathrm{u}^{(i)} - L, \mathrm{u}^{(i)} + L\right) \subset \mathbb{R}^d$$

and the ball in $\mathbb{R}^{nd}$ of radius L centered at $\mathbf{u} = (u_1, \dots, u_N)$ is the cube

$$\boldsymbol{\Lambda}_L(\mathbf{u}) = \mathop{\times}_{j=1}^{n} \Lambda_L(u_j) \subset \mathbb{R}^{nd}. \tag{1.4}$$

Sometimes we will use the symbol $\boldsymbol{\Lambda}_L^{(n)}(\mathbf{u})$ to put emphasis on the number of particles in the system. For our purposes, it suffices to consider only cubes centered at lattice points $u \in \mathbb{Z}^d$ and $\mathbf{u} \in \mathbb{Z}^{nd}$. For that reason, letters u, v, w and $\mathbf{u}, \mathbf{v}, \mathbf{w}$ will always refer to points in the corresponding lattices.

We denote by $\mathbf{1_A}$ the characteristic function of a set $\mathbf{A} \subset \mathbb{R}^{nd}$ and also, with a standard abuse of notation, the operator of multiplication by this function.

We also need "lattice cubes":

$$B_L(u) = \Lambda_L(u) \cap \mathbb{Z}^d, \quad \mathbf{B}_L(\mathbf{u}) = \boldsymbol{\Lambda}_L(\mathbf{u}) \cap \mathbb{Z}^{nd}, \tag{1.5}$$

and "unit cells", or simply "cells":

$$C(u) = \Lambda_1(u) \subset \mathbb{R}^d, \quad \mathbf{C}(\mathbf{u}) = \boldsymbol{\Lambda}_1(\mathbf{u}) \subset \mathbb{R}^{nd}. \tag{1.6}$$

In what follows, all these sets are often called "boxes", single-particle boxes for $\Lambda_L(u)$, $B_L(u)$ and $C(u)$ and n-particle boxes for $\boldsymbol{\Lambda}_L(\mathbf{u})$, $\mathbf{B}_L(\mathbf{u})$ and $\mathbf{C}(\mathbf{u})$. A "cellular set" is a finite union of cells.

We will also use "annular" sets, or shortly, annuli, defined as the difference $\boldsymbol{\Lambda}_{L+\mathrm{w}}(\mathbf{u})\setminus \boldsymbol{\Lambda}_L(\mathbf{u})$ where $\mathrm{w}>0$ is the width and $\mathbf{u}$ the centre.

1.3. **Interaction potential.** The interaction potential U is of the form

$$U(\mathbf{x}) = \sum_{k=1}^{N} \sum_{1\le i_1<\cdots<i_k\le N} \Phi^{(k)}(x_{i_1},\dots,x_{i_k}) \tag{1.7}$$

where $\mathbf{x}=(x_1,\dots,x_N)\in\mathbb{R}^{Nd}$. The functions $\Phi^{(k)}\colon \mathbb{R}^{d\times k}\to\mathbb{R}$ are k-body interaction potentials, $k=1,\dots,N$, satisfying the following properties **(I 1)**-**(I 2)**, for $k=1,\dots,N$ and $\mathbf{y}=(y_1,\dots,y_k)\in\mathbb{R}^{d\times k}$:

(I 1) *Boundedness and nonnegativity*: There exists a constant $\mathrm{u}_0\in(0,+\infty)$ such that

$$0\le \Phi^{(k)}(\mathbf{y})\le \mathrm{u}_0. \tag{1.8}$$

(I 2) *Finite range*: For some constant $\mathrm{r}_0\in(0,+\infty)$ and for $k=2,\dots,N$,

$$\max_{1\le i\le k}\min_{j\ne i}|y_i-y_j|\ge \mathrm{r}_0 \implies \Phi^{(k)}(\mathbf{y})=0. \tag{1.9}$$

Remark 1.1. The non-negativity of the potentials $\Phi^{(k)}$ is used to simplify the statement of the main result (see Theorem 1.5 below) and shorten the proof of technical assertions.

We can also relax the boundedness condition, by allowing "hard-core potentials", such that, for any $k=2,\dots,N$, and for $0<\mathrm{r}_1<\mathrm{r}_0$,

$$\min_{1\le i<j\le k}|y_i-y_j|<\mathrm{r}_1 \implies \Phi^{(k)}(\mathbf{y})=+\infty.$$

While symmetry of the interaction is not important for our methods, it is usually assumed in physical applications.

On the other hand, the finite-range condition is essential. Extending Theorem 1.5 to the case of infinite-range potentials seems an important and challenging problem.

1.4. **External field potential.** As mentioned before, the random external potential $V(x;\omega)$, $x\in\mathbb{R}^d$, $\omega\in\Omega$, is assumed to be of alloy-type, over a cubic lattice. That is,

$$V(x;\omega)=\sum_{s\in\mathbb{Z}^d}\mathrm{V}_s(\omega)\varphi_s(x-s). \tag{1.10}$$

Here $\{\mathrm{V}_s\}_{s\in\mathbb{Z}^d}$, is a family of IID (independent, identically distributed) real random variables V_s on some probability space $(\Omega,\mathfrak{B},\mathbb{P})$ and $\{\varphi_s\}_{s\in\mathbb{Z}^d}$ is a (nonrandom) collection of "bump" functions (not necessarily identical)

$$\mathbb{R}^d\ni y\mapsto \varphi_s(y).$$

In probabilistic terms, V is a real-valued random field (RF) on $\mathbb{Z}^d$. Physically speaking, random variable V_s represents the amplitude of the "impurity" at the site $s\in\mathbb{Z}^d$ while function φ_s describes the "propagation" of the impact of this impurity across $\mathbb{R}^d$.

1.4.1. We assume the following conditions **(E 1)**-**(E 2)**.

(E 1) *Boundedness and nonnegativity*:

$$\operatorname{ess\,sup}\ \mathrm{V}_s<\infty,\quad \operatorname{ess\,inf}\ \mathrm{V}_s=0. \tag{1.11}$$

Remark 1.2. Again, the nonnegativity plays a technical role and is not crucial for the main result. The boundedness condition can be replaced by finiteness of moments $\mathbb{E}|\mathrm{V}_s|^{\mathrm{r}}$ for some $\mathrm{r}>0$.

Let $s \in \mathbb{Z}^d$ be a given site. Consider the distribution function:

$$F(\mathrm{y}) := \mathbb{P}(\mathrm{V}_s < \mathrm{y}), \quad \mathrm{y} \in \mathbb{R}, \tag{1.12}$$

Condition (E1) implies that $F(\mathrm{y}) = 0$ for $\mathrm{y} < 0$ and $F(\mathrm{y}) = 1$ for y large enough.

(E 2) *Uniform Hölder-continuity of* $F(\mathrm{y}|\mathfrak{B}_s^{\mathrm{c}})$: There exist constants a, $b > 0$ such that for all $\epsilon \in (0, 1)$,

$$\nu(\epsilon) := \sup_{s\in\mathbb{Z}^d} \sup_{\mathrm{y}\in\mathbb{R}} \big[F(\mathrm{y} + \epsilon) - F(\mathrm{y})\big] \le a\epsilon^b. \tag{1.13}$$

Remark 1.3. The main result of this paper remains valid under a weaker assumption of log-Hölder continuity: $\nu(\epsilon) \le a|\ln\epsilon|^{-b}$, for $b > 0$ large enough.

1.4.2. Lastly, we require two more conditions, **(E 3)-(E 4)**, on bump functions φ_s.

(E 3) *Boundedness, nonnegativity and compact support of* φ_s: Functions φ_s are nonnegative and have a compact support: $\mathrm{diam}\,(\mathrm{supp}\,\varphi_s) \le R$, so that

$$\sup_{x\in\mathbb{R}^d} \sum_{s\in\mathbb{Z}^d} \varphi_s(x - s) < +\infty, \tag{1.14}$$

(E 4) *Covering condition for* φ_s: For all $L \ge 1$, $u \in \mathbb{R}^d$ and $x \in \Lambda_L(u)$,

$$\sum_{s\in\Lambda_L(u)\cap\mathbb{Z}^d} \varphi_s(x - s) \ge 1. \tag{1.15}$$

Remark 1.4. As above, assumptions **(E 3)-(E 4)** can be relaxed.

From now on we assume that values $d \ge 1$ and $N > 1$ are fixed. We will work with fixed interaction potentials $\Phi^{(k)}$ in Eqn (1.7), $1 \le k \le N$, a fixed collection of bump functions φ_s from Eqn (1.10) and a fixed distribution function F in Eqn (1.12), assuming the conditions **(I 1)-(I 2)** and **(E 1)-(E 4)**.

1.5. **Main result.** Under conditions **(I 1)-(I 2)** and **(E 1)-(E 4)**, operator $\mathbf{H}^{(N)}(\omega)$ is correctly defined for $\mathbb{P}$-almost all $\omega \in \Omega$ (as a unique self-adjoint extension from the set of C^2-functions $f(\mathbf{x})$ with compact support). Furthermore, the (nonrandom) operator

$$\mathbf{H}_0^{(N)} = -\frac{1}{2}\mathbf{\Delta} + \mathbf{U}$$

is also correctly defined and has the lower edge of its spectrum at a point $E^0 \ge 0$.

Theorem 1.5. *Let* $\mathbf{H}^{(N)}(\omega)$ *be the random operator defined in Eqn* (1.1). *Then* $\exists$ *nonrandom constants* $\eta^* > 0$ *and* $m^* > 0$ *such that, with* $\mathbb{P}$*-probability one,*

(i) *The spectrum of* $\mathbf{H}^{(N)}(\omega)$ *in* $[E^0, E^0 + \eta^*]$ *is non-empty and pure point.*

(ii) *All eigenfunctions* $\boldsymbol{\Psi}_j(\mathbf{x};\omega)$ *of* $\mathbf{H}^{(N)}(\omega)$ *with eigenvalues* $E_j(\omega) \in [E^0, E^0 + \eta^*]$ *satisfy exponential bounds*

$$\big\|\mathbf{1}_{\mathbf{C}(\mathbf{u})}\boldsymbol{\Psi}_j(\,\cdot\,;\omega)\big\|_{L_2(\mathbb{R}^{Nd})} \le c_j(\omega)\mathrm{e}^{-m^*|\mathbf{u}|}, \quad \mathbf{u} \in \mathbb{Z}^{Nd}, \tag{1.16}$$

where $c_j(\omega) \in (0, +\infty)$ *are random constants.*

A direct application of general results on local regularity of (generalized) eigenfunctions of Schrödinger operators, (see, e.g., [6]), gives rise to the following

Corollary 1.6. *The eigenfunctions $\boldsymbol{\Psi}_j(\mathbf{x};\omega)$ with eigenvalues $E^0 \le E_j(\omega) \le E^0 + \eta^*$ satisfy the bounds:*

$$|\boldsymbol{\Psi}_j(\mathbf{x};\omega)| \le \widetilde{c}_j(\omega)\, \mathrm{e}^{-\widetilde{m}^*|\mathbf{x}|},\ \ \mathbf{x}\in\mathbb{R}^{Nd}, \tag{1.17}$$

with $\widetilde{m}^* > 0$ and random constants $\widetilde{c}_j(\omega)\in(0,+\infty)$.

Remark 1.7. Theorem 1.5 addresses the spectrum of $\mathbf{H}^{(N)}(\omega)$ in the whole Hilbert space $L_2(\mathbb{R}^{Nd})$. This, of course, covers subspaces $L_2^{\mathrm{sym}}(\mathbb{R}^{Nd})$ and $L_2^{\mathrm{asym}}(\mathbb{R}^{Nd})$ formed by symmetric and antisymmetric functions (bosonic and fermionic subspaces, respectively).

Next, as explained in Section 4, when we increase the number of particles N, keeping fixed the structure of potentials U and V, the width $\eta^* \to 0$. A similar phenomenon is observed (due to essentially similar reasons) when one uses the MSA to prove pure point spectrum in a single-particle Anderson model with growing dimension d of the one-particle configuration space $\mathbb{R}^d$. In fact, key estimates of the single-particle MSA cannot be made uniform in d, in the framework of existing technical tools. On the other hand, it is possiblle to modify the argument presented in this paper and show that if the external random potential field has the form $gV(x,\omega)$, $g>0$ being a coupling amplitude, then the width η^* can be made of order $O(g)$, for any given value of N.

1.6. **From MSA bounds to dynamical localization.** The derivation of the dynamical localization from sufficiently strong MSA bounds in the framework of single-particle Anderson models, on a lattice or in a Euclidean space, is well-understood by now. For the multi-particle model considered in this paper, the derivation of dynamical localization from the key MSA bounds proven below in sections 4–7 requires only a few minor modifications of known techniques, essentially of geometrical nature. We plan to publish it in a separate paper, in order to keep the size of the present manuscript within reasonable limits.

1.7. **On the multi-particle MSA.** In the proof of Theorem 1.5 we will focus on properties of finite-volume versions $\mathbf{H}_{\boldsymbol{\Lambda}} = \mathbf{H}_{\boldsymbol{\Lambda}}^{(N)}(\omega)$ of Hamiltonian $\mathbf{H}$. More precisely, let $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}^{(N)}(\mathbf{u})$ be an N-particle box and consider the operator $\mathbf{H}_{\boldsymbol{\Lambda}}^{(N)}(\omega)$ in $L^2(\boldsymbol{\Lambda})$, (referred to as the Hamiltonian of the N-particle system in $\boldsymbol{\Lambda}$) of the same structure as in (1.1), (1.7) and (1.10):

$$\mathbf{H}_{\boldsymbol{\Lambda}}^{(N)}(\omega) = -\frac{1}{2}\boldsymbol{\Delta}^{\boldsymbol{\Lambda}} + \mathbf{U} + \mathbf{V}(\omega). \tag{1.18}$$

Here $\boldsymbol{\Delta}^{\boldsymbol{\Lambda}}$ stands for the Laplacian in $\boldsymbol{\Lambda}$ with Dirichlet's boundary conditions on $\partial\,\boldsymbol{\Lambda}$.

The spectrum of a given operator, e.g., $\mathbf{H}$, will be denoted as $\sigma(\mathbf{H})$.

Under conditions **(I 1)-(I 2)** and **(E 1)-(E 4)**, for for $\mathbb{P}$-almost all $\omega\in\Omega$, operator $\mathbf{H}_{\boldsymbol{\Lambda}}^{(N)}(\omega)$ is correctly defined in $L_2(\boldsymbol{\Lambda})$, as a unique self-adjoint extension from the domain $C_0^2(\boldsymbol{\Lambda})$. Moreover, $\mathbf{H}_{\boldsymbol{\Lambda}}^{(N)}(\omega)$ has a discrete spectrum, since its resolvent

$$\mathbf{G}^{\boldsymbol{\Lambda}}(E) = \big(\mathbf{H}_{\boldsymbol{\Lambda}} - E\big)^{-1},\ \ \text{for } E\in\mathbb{R}\setminus\sigma\big(\mathbf{H}_{\boldsymbol{\Lambda}}\big), \tag{1.19}$$

is a compact integral operator; it will be in the centre of our attention. Its kernel

$$\boldsymbol{\Lambda}\times\boldsymbol{\Lambda}\ni(\mathbf{x},\mathbf{x}')\mapsto \mathrm{G}^{\boldsymbol{\Lambda}}(\mathbf{x},\mathbf{x}';E),\ \ \mathbf{x},\mathbf{x}'\in\boldsymbol{\Lambda}, \tag{1.20}$$

is known as the Green function of $\mathbf{H}_{\boldsymbol{\Lambda}}$. The MSA is based on an asymptotical analysis of resolvent $\mathbf{G}^{\boldsymbol{\Lambda}}(E)$ as $\boldsymbol{\Lambda}\nearrow\mathbb{R}^{Nd}$. More precisely, boxes $\boldsymbol{\Lambda}$ will have the form

$$\boldsymbol{\Lambda} = \boldsymbol{\Lambda}_{L_k}(\mathbf{u}),\ \mathbf{u}\in\mathbb{Z}^{Nd},\ k=0,1,\dots,$$

where positive integers L_k are determined by a recurrence involving a starting value L_0 and a number $\alpha > 1$:

$$L_k = [L_{k-1}^{\alpha}] \sim (L_0)^{\alpha^k},\ k \geq 1. \tag{1.21}$$

Here $[\cdot]$ stands for the integer part. In future, we will take $\alpha = 3/2$. Nevertheless, to keep a connection with the literary tradition, we will continue using symbol α. The same can be said about parameter $\beta > 0$ appearing in (3.3): its value will be $\beta = 1/2$.

The positive integer value L_0 (the radius of box $\boldsymbol{\Lambda}_{L_0}(\mathbf{u})$) will be eventually assumed to be large enough (depending on technical constants emerging in the course of our argument, which, in turn, are determined by d, N, $\{\Phi^{(k)}\}$, F and $\{\varphi_s\}$; cf. (1.7), (1.10) and (1.12)). However, in several definitions and related constructions the value of L_0 will only have to satisfy some trivial restrictions, obvious from the context.

To put it simply, one needs L to be large enough for the asymptotic relations of the form $\ln L \ll L^a \ll e^{L^\beta} \ll e^{bL}$ (with $a, b > 0, \beta \in (0,1)$) to hold.

Summarising, for future references,

$$\alpha = \frac{3}{2}, \quad \beta = \frac{1}{2}, \quad L_0 \text{ is a positive integer, large enough.} \tag{1.22}$$

To the reader familiar with the MSA method in localisation proofs we can say at this stage that the existence of values $\eta^* > 0$, $m^* > 0$ and $p^* > Nd$ claimed in Theorem 1.13 below will emerge as a result of a 'combined' induction, in the number of particles, N, and the 'scaling' index k appearing in in (1.21).

Consequently, in the course of the argument, we will often work with n-particle Hamiltonians $\mathbf{H}_{\boldsymbol{\Lambda}}^{(n)}(\omega)$, of the same form as in (1.18), with $n = 1, \ldots, N$. *Mutatis mutandis*, definitions and facts introduced/noted for an N particle system will be used for a system of n particles as well. (In fact, some technical constructions will be carried on for an n-particle system first, and then put in the context of n running through the values $n = 1, \ldots, N$).

Concluding this subsection, we stress that all eigenvectors of finite-volume Hamiltonians appearing in our arguments and calculations are normalised.

1.8. **Separable boxes and MSA estimates.** The principal difficulty encountered while attempting to extend existing sigle-particle methods of localization theory to multi-particle Anderson models arises from the (innocently looking) summatory formula $\mathbf{x} \in \mathbb{R}^{Nd} \mapsto \sum_{i=1}^{N} V(x_i;\omega)$ for the external potential in Eqn (1.2). In our context, the values of the N-particle external potential exhibit, for various points $\mathbf{x}$, infinite-range correlations. For example, suppose that vectors $\mathbf{x} = (x_1, \ldots, x_N)$ and $\mathbf{x}' = (x'_1, \ldots, x'_N)$ include components x_j and x'_j with $\varphi_s(x_j - s)\varphi_s(x'_j - s) \neq 0$ for some $s \in \mathbb{Z}$ (which physically means that the distance $|x_j - x'|$ is small). Then the random variable V_s will be present in both sums $\sum_{i=1}^{N} V(x_i;\omega)$ and $\sum_{i=1}^{N} V(x'_i;\omega)$, generating their dependence on each other.

This difficulty has been overcame in an analysis of regularity of the so-called density of states (cf. [13]) and of eigenvalue distribution of finite-volume multi-particle Hamiltonians (cf. [12]). Unfortunately, the information on regularity of the distribution of eigenvalues in any given multi-particle box $\boldsymbol{\Lambda}_L(\mathbf{x})$ does not provide a sufficient input for the MSA. At the same time, the existence of the multi-particle density of states is not required *per se* for the MSA to work.

We tackle this issue by using the concept of separability of boxes, which figures explicitly in Theorems 1.12 and 1.13 below. More precisely, it is the so-called Wegner-type bound $\mathbf{W2}(n)$ that requires the notion of separability (see Eqn (4.2)). We want to stress here (as we did in [4]) that the MSA requires Wegner-type bounds of two types: **(i)** for one multi-particle box $\boldsymbol{\Lambda}$ and **(ii)** for two multi-particle boxes $\boldsymbol{\Lambda}$ and $\boldsymbol{\Lambda}'$. See bounds $\mathbf{W1}(n)$ and $\mathbf{W2}(n)$ in Eqns (4.1) and (4.2) below. However, the MSA is less sensitive to optimality in these bounds (which may be important for other areas in physics of disordered systems). For the first time this notion has been used, in the context of a two-particle lattice Anderson model, in [7] and [8]. The extension to the N-particle lattice case was carried out in [9].

We now turn to the formal aspect of separability. Given an n-particle box $\boldsymbol{\Lambda}_L^{(n)}(\mathbf{u}) \subset \mathbb{R}^{nd}$ and $j = 1, \dots, n$, denote by $\Pi_j\, \boldsymbol{\Lambda}^{(n)}(\mathbf{u}) \subset \mathbb{R}^d$ the projection of $\boldsymbol{\Lambda}^{(n)}(\mathbf{u})$ to the jth factor in $\mathbb{R}^{nd}$: if $\boldsymbol{\Lambda}_L(\mathbf{u}) = \prod_{i=1}^{n} \Lambda_L(u_i)$ then $\Pi_j\, \boldsymbol{\Lambda}^{(n)}(\mathbf{u}) = \Lambda_L(u_i)$. Further, define the 'full projection' $\Pi\, \boldsymbol{\Lambda}^{(n)}(\mathbf{u})$ of $\boldsymbol{\Lambda}^{(n)}(\mathbf{u})$:

$$\Pi\, \boldsymbol{\Lambda}^{(n)}(\mathbf{u}) := \bigcup_{j=1}^{n} \Pi_j\, \boldsymbol{\Lambda}^{(n)}(\mathbf{u}) \subset \mathbb{R}^d.$$

Definition 1.8. Let $n = 1, \dots, N$ and assume $\mathcal{J}$ is a non-empty subset in $\{1, \dots, n\}$. We say that a box $\boldsymbol{\Lambda}_L^{(n)}(\mathbf{y})$ is $\mathcal{J}$-separable from box $\boldsymbol{\Lambda}_L^{(n)}(\mathbf{x})$ if

$$\Big(\bigcup_{j\in\mathcal{J}} \Pi_j\, \boldsymbol{\Lambda}_{L+R}^{(n)}(\mathbf{y})\Big) \bigcap \Big(\bigcup_{i\notin\mathcal{J}} \Pi_i\, \boldsymbol{\Lambda}_{L+R}^{(n)}(\mathbf{y}) \bigcup \Pi\, \boldsymbol{\Lambda}_{L+R}^{(n)}(\mathbf{x})\Big) = \varnothing, \tag{1.23}$$

where R is the constant from condition **(E 3)**.

Next, a pair of boxes $\boldsymbol{\Lambda}_L^{(n)}(\mathbf{x})$, $\boldsymbol{\Lambda}_L^{(n)}(\mathbf{y})$ is said to be *separable* if, for some non-empty set $\mathcal{J} \subset \{1, \dots, n\}$, $\operatorname{dist}\Big(\boldsymbol{\Lambda}_L^{(n)}(\mathbf{x}), \boldsymbol{\Lambda}_L^{(n)}(\mathbf{y})\Big) > 2N(L+R)$ and

- either $\boldsymbol{\Lambda}_L^{(n)}(\mathbf{y})$ is $\mathcal{J}$-separable from $\boldsymbol{\Lambda}_L^{(n)}(\mathbf{x})$,
- or $\boldsymbol{\Lambda}_L^{(n)}(\mathbf{x})$ is $\mathcal{J}$-separable from $\boldsymbol{\Lambda}_L^{(n)}(\mathbf{y})$.

In physical terms: let box $\boldsymbol{\Lambda}_L^{(n)}(\mathbf{x})$ be $\mathcal{J}$-separable from $\boldsymbol{\Lambda}_L^{(n)}(\mathbf{y})$ and consider two quantum n-particle systems, in $\boldsymbol{\Lambda}_L^{(n)}(\mathbf{x})$ and $\boldsymbol{\Lambda}_L^{(n)}(\mathbf{y})$ (i.e., with Hamiltonians $\mathbf{H}^{\boldsymbol{\Lambda}_L^{(n)}(\mathbf{x})}$ and $\mathbf{H}^{\boldsymbol{\Lambda}_L^{(n)}(\mathbf{y})}$). Then the first system contains a 'detached' subsystem, formed by particles with labels from $\mathcal{J}$, with the following property. $\forall\ \mathbf{u} = (u_1, \dots, u_n) \in \boldsymbol{\Lambda}_L^{(n)}(\mathbf{x})$ and $\mathbf{v} = (v_1, \dots, v_n) \in \boldsymbol{\Lambda}_L^{(n)}(\mathbf{y})$, the collection of random variables V_s from RF $\mathcal{V}$ contributing into the external potential sum $\sum_{j\in\mathcal{J}} V(x_j;\omega)$ is disjoint from similarly defined collections, for sums $\sum_{j\notin\mathcal{J}} V(x_j;\omega)$ and $\sum_{1\le j\le n} V(x'_j;\omega)$. This implies independence of sum $\sum_{j\in\mathcal{J}} V(x_j;\omega)$ and the pair of sums $\sum_{j\notin\mathcal{J}} V(x_j;\omega)$ and $\sum_{1\le j\le n} V(x'_j;\omega)$ and provides enough 'randomness' to produce satisfactory estimates.

Lemma 1.9. *Given $n \geq 2$, set $\kappa = \kappa(n) = n^n$. For any $L > 1$ and n-particle configuration $\mathbf{x} \in \mathbb{Z}^{nd}$, there exists a collection of n-particle boxes $\boldsymbol{\Lambda}_{L^{(l)}}(\mathbf{x}^{(l)})$, $l = 1, \dots, K(\mathbf{x}, n)$, with $K(\mathbf{x}, n) \leq \kappa$ and $L^{(l)} \leq 2n(L+R)$, such that if a vector $\mathbf{y} \in \mathbb{Z}^{nd}$*

satisfies

$$\mathbf{y} \notin \bigcup_{\ell=1}^{K(\mathbf{x},n)} \boldsymbol{\Lambda}_{L^{(l)}}(\mathbf{x}^{(l)}), \tag{1.24}$$

then boxes $\boldsymbol{\Lambda}_L^{(n)}(\mathbf{x})$ *and* $\boldsymbol{\Lambda}_L^{(n)}(\mathbf{y})$ *with* dist $\left(\boldsymbol{\Lambda}_L^{(n)}(\mathbf{x}), \boldsymbol{\Lambda}_L^{(n)}(\mathbf{y})\right) > 2N(L+R)$ *are separable.*

In particular, a pair of boxes $\boldsymbol{\Lambda}_L^{(n)}(\mathbf{x})$, $\boldsymbol{\Lambda}_L^{(n)}(\mathbf{y})$ *with* dist $\left(\boldsymbol{\Lambda}_L^{(n)}(\mathbf{x}), \boldsymbol{\Lambda}_L^{(n)}(\mathbf{y})\right) > 2NL$ *is separable if*

$$\boldsymbol{\Lambda}_{L+R}^{(n)}(\mathbf{y}) \bigcap \boldsymbol{\Lambda}_{|\mathbf{x}|+L+R}^{(n)}(\mathbf{0}) = \varnothing.$$

For the proof of Lemma 1.9, see Section 8.

Corollary 1.10. *Fix two integers,* $n \geq 2$ *and* $L > 1$*, and let* $\kappa < \infty$ *be the number defined in Lemma* 1.9*. Set* $B = 4n(L+R)+1$ *and consider an* n*-particle box* $\boldsymbol{\Lambda}_L(\mathbf{x})$ *and* $2\kappa+1$ *disjoint concentric annular sets* $\boldsymbol{A}_1(\mathbf{x})$, ..., $\boldsymbol{A}_{2\kappa+1}$ *around* $\boldsymbol{\Lambda}_L(\mathbf{x})$*:*

$$\boldsymbol{A}_j(\mathbf{x}) = \boldsymbol{\Lambda}_{L+jB}(\mathbf{x}) \setminus \boldsymbol{\Lambda}_{L+(j-1)B}(\mathbf{x}), \quad j = 1, \ldots, 2\kappa+1.$$

Then at least one of the annuli $\boldsymbol{A}_{2j-1}(\mathbf{x})$, $1 \leq j \leq \kappa+1$*, contains no box* $\boldsymbol{\Lambda}_L(\mathbf{y})$ *not separable from* $\boldsymbol{\Lambda}_L(\mathbf{x})$*.*

Proof. Assume otherwise and consider $\kappa+1$ boxes $\boldsymbol{\Lambda}_L(\mathbf{y}_j) \subset \boldsymbol{A}_{2j-1}(\mathbf{x})$, $j = 1, \ldots, \kappa+1$, which are not separable from $\boldsymbol{\Lambda}_L(\mathbf{x})$. Since

$$\operatorname{dist}(\boldsymbol{\Lambda}_L(\mathbf{y}_j), \boldsymbol{\Lambda}_L(\mathbf{y}_{j+1})) \geq \operatorname{dist}(\boldsymbol{A}_j(\mathbf{x}), \boldsymbol{A}_{j+1}(\mathbf{x})) - 2(L+R) > 4n(L+R),$$

these $\kappa(n)+1$ boxes cannot be enclosed in $\kappa(n)$ boxes of radius $2n(L+R)$, in contradiction to the first assertion of Lemma 1.9. ☐

We would like to stress that

- the value κ depends only upon the number of particles n;
- in the case where boxes $\boldsymbol{\Lambda}_L^{(n)}(\mathbf{x})$ and $\boldsymbol{\Lambda}_L^{(n)}(\mathbf{0})$ are disjoint, it is always true that box $\boldsymbol{\Lambda}_L^{(n)}(\mathbf{x})$ is $\mathcal{J}$-separable from $\boldsymbol{\Lambda}_L^{(n)}(\mathbf{x})$, for some $\mathcal{J} \subseteq \{1, \ldots, n\}$.

Define the outer layer (of width 2) in a box $\boldsymbol{\Lambda}_L(\mathbf{u})$ and its lattice counterpart $\mathbf{B}_L(\mathbf{u})$:

$$\boldsymbol{\Lambda}_L^{\text{out}}(\mathbf{u}) = \boldsymbol{\Lambda}_L(\mathbf{u}) \setminus \boldsymbol{\Lambda}_{L-2}(\mathbf{u}), \;\; \mathbf{B}_L^{\text{out}}(\mathbf{u}) = \boldsymbol{\Lambda}_L^{\text{out}}(\mathbf{u}) \cap \mathbb{Z}^{nd}, \;\; \mathbf{u} \in \mathbb{Z}^{nd}. \tag{1.25}$$

For given $m > 0$ and $L \geq 1$, set :

$$\gamma(m, L, n) \; (\, = \gamma_N(m, L, n)) \;\; = mL\left(1 + L^{-1/4}\right)^{N-n+1}, \; 1 \leq n \leq N. \tag{1.26}$$

Definition 1.11 ((E, m)-nonsingularity)**.** Let $E \in \mathbb{R}$ and $m > 0$. We say that box $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}_L^{(n)}(\mathbf{u}) \subset \mathbb{R}^{nd}$, $1 \leq n \leq N$, is (E, m)-*nonsingular* ((E, m)-NS) if $E \in \mathbb{R} \setminus \sigma(\mathbf{H}_{\boldsymbol{\Lambda}})$ and for any $\mathbf{y} \in \mathbf{B}_L^{\text{out}}(\mathbf{u})$, the L_2-norm of the operator $\mathbf{1}_{\mathbf{C}(\mathbf{u})}\, \mathbf{G}^{\boldsymbol{\Lambda}}(E)\, \mathbf{1}_{\mathbf{C}(\mathbf{y})}$ satisfies the bound

$$\| \, \mathbf{1}_{\mathbf{C}(\mathbf{u})}\, \mathbf{G}^{\boldsymbol{\Lambda}}(E)\, \mathbf{1}_{\mathbf{C}(\mathbf{y})} \, \|_{L_2(\mathbb{R}^{Nd})} \leq \mathrm{e}^{-\gamma(m,L,n)}. \tag{1.27}$$

Otherwise, $\boldsymbol{\Lambda}$ is called (E, m)-*singular* ((E, m)-S).

Similarly, a lattice box $\mathbf{B}_L(\mathbf{u}) = \boldsymbol{\Lambda}_L(\mathbf{u}) \cap \mathbb{Z}^{nd}$ is called (E, m)-NS or (E, m)-S when the Euclidean box $\boldsymbol{\Lambda}_L(\mathbf{u})$ is (E, m)-NS or (E, m)-S, respectively.

Consider the following property:

DS(m,p,k,I,n): Given $m>0$, $k=0,1,\dots$ and an interval $I\subseteq\mathbb{R}$, for any pair of separable boxes $\boldsymbol{\Lambda}^{(n)}_{L_k}(\mathbf{u})$, $\boldsymbol{\Lambda}^{(n)}_{L_k}(\mathbf{v})$, the probability

$$\mathbb{P}\big\{\ \forall\ \ E\in I,\ \boldsymbol{\Lambda}^{(n)}_{L_k}(\mathbf{u})\text{ or }\boldsymbol{\Lambda}^{(n)}_{L_k}(\mathbf{v})\text{ is }\ (E,m)\text{–NS}\ \big\}\geq 1-L_k^{-2p}. \tag{1.28}$$

Recall: L_k stands for an integer of the form (1.21), with α as in (1.22). The abbreviation DS means 'double singularity'.

Property **DS**(m,p,k,I,N) (with $n=N$), is critical for the N-particle MSA scheme; see Theorem 1.12 below. Once this property is established for all $k\geq 0$ (at the end of Section 7), it will mark the end of the proof of Theorem 1.5.

Theorem 1.12. *Let $I\subseteq\mathbb{R}$ be an interval. Assume that for some $m>0$, $L_0>2$, $p>Nd$ and for any $k\geq 0$, property* **DS**(m,p,k,I,N) *holds true, with L_k as in Eqns* (1.21)*,* (1.22)*.*

Then, with $\mathbb{P}$-probability one,

(i) *The spectrum of $\mathbf{H}^{(N)}(\omega)$ in I is pure point.*

(ii) *The eigenfunctions $\boldsymbol{\Psi}_j(\mathbf{x};\omega)$ of Hamiltonian $\mathbf{H}^{(N)}(\omega)$ with eigenvalues $E_j(\omega)\in I$ satisfy the exponential bounds similar to Eqn* (1.16)*:*

$$\big\|\mathbf{1}_{\mathbf{C}(\mathbf{u})}\boldsymbol{\Psi}_j(\ \cdot\ ;\omega)\big\|_{L_2(\mathbb{R}^{Nd})}\leq c_j(\omega)\mathrm{e}^{-m|\mathbf{u}|},\ \ \mathbf{u}\in\mathbb{Z}^{Nd}. \tag{1.29}$$

Theorem 1.12 represents an 'analytic' part of the MSA. (Probability plays a subordinate role here, reduced merely to the Borel-Cantelli lemma, which is guaranteed by the fact that $p>Nd$.) The proof of Theorem 1.12 is 'standard', in the sense that it does not use particulars of the model involved. We therefore omit the proof of Theorem 1.12 from the paper, referring the reader to [7, Theorem 2] and [9, Theorem 2]. (In fact, the proof of Theorem 1.12 follows almost *verbatim* the proof of Theorem 2.3 from [11].)

In view of Theorem 1.12, the assertion of Theorem 1.5 can be deduced from the following Theorem 1.13.

Theorem 1.13. *Under assumptions of Theorem 1.5, there exist $\eta^*>0$ sufficiently small, $p^*>Nd$, and $m^*>0$ such that, for an integer $L_0>1$ large enough, property* **DS**(m,p,k,I,N) *holds for all $k\geq 0$, with $p=p^*$, $m=m^*$, $I=[E^0,E^0+\eta^*]$ and L_k as in Eqns* (1.21)*,* (1.22)*.*

The rest of the paper is devoted to the proof of Theorem 1.13. This theorem represents a 'probabilistic' part of the MSA; unlike Theorem 1.12, its proof is quite sensitive to particulars of a given model. Nevertheless, we will follow the same logical scheme as in [9, Theorem 3].

1.9. Comments on the structure of the paper.

- In Section 2, we adapt well-known "geometric resolvent inequalities", established for Schrödinger operators in Euclidean spaces. As a result, we state these inequalities in a form convenient for subsequent analysis of the above norm $\|\,\mathbf{1}_{\mathbf{C}(\mathbf{u})}\,\mathbf{G}^{\boldsymbol{\Lambda}_L(\mathbf{v})}(E)\,\mathbf{1}_{\mathbf{C}(\mathbf{w})}\,\|$.
- In Section 3, following Ref. [5], we discuss a useful notion of 'lattice subharmonicity'. It is subsequently used in Sections 6 and 7. A reader familiar with the MSA may favour a different argument while proving the main result of Section 3, Lemma 3.1 (cf., e.g, the proof of Lemma 4.2 in [11]) and skip the rest of Section 3.

- Further, Section 4 describes the MSA inductive scheme adopted in this paper and establishes the initial step of the induction. We then discuss the structure of the argument in the inductive step. To conduct the inductive step, we have to analyse three types of pairs of separable boxes $\boldsymbol{\Lambda}^{(n)}_{L_k}(\mathbf{u})$, $\boldsymbol{\Lambda}^{(n)}_{L_k}(\mathbf{v})$ figuring in Eqn (1.28). These types are described as partially interactive, fully interactive or mixed pairs $\boldsymbol{\Lambda}^{(n)}_{L_k}(\mathbf{u})$, $\boldsymbol{\Lambda}^{(n)}_{L_k}(\mathbf{v})$, depending on a property of 'decomposability' of the corresponding particle systems into non-interacting subsystems.
- In Sections 5–7 we give a case-by-case analysis of each of the three aforementioned types. Sections 4–7 are in fact adaptations, for multi-particle alloy-type Anderson models, of the argument from Sections 4–7 of paper [9] where the focus was on multi-particle lattice Anderson models. Here we systematically refer to various results and techniques for Schrödinger operators in a Euclidean space, summarised and developed in the monograph [14].
- Section 8 is an appendix containing (elementary) proofs of two basic (but convenient) facts used in the main body of the paper.

2. Resolvent inequalities

Throughout Sections 2-3, we work with a fixed bounded interval $I \subset \mathbb{R}$ and variable $n = 1, ..., N$.

2.1. **Geometric resolvent inequality.** Given an n-particle box $\boldsymbol{\Lambda}_L(\mathbf{u}) \subset \mathbb{R}^{nd}$ with $L \geq 4$, we define the interior $\boldsymbol{\Lambda}^{\rm int}_L(\mathbf{u})$ of $\boldsymbol{\Lambda}_L(\mathbf{u})$ by

$$\boldsymbol{\Lambda}^{\rm int}_L(\mathbf{u}) = \boldsymbol{\Lambda}_{\lceil L/3 \rceil}(\mathbf{u}). \tag{2.1}$$

Next, consider two n-particle boxes, $\boldsymbol{\Lambda}_L(\mathbf{u}) \subset \boldsymbol{\Lambda}_{\widetilde{L}}(\mathbf{u})$, with $4 \leq L < \widetilde{L}$, and cellular subsets

$$\mathbb{A} \subset \boldsymbol{\Lambda}^{\rm int}_L(\mathbf{u}) \;\text{ and }\; \mathbb{B} \subset \boldsymbol{\Lambda}_{\widetilde{L}}(\mathbf{u}) \setminus \boldsymbol{\Lambda}_L(\mathbf{u}).$$

From now on we will omit subscript $L_2(\mathbb{R}^{nd})$ in the notation $\| \cdot \|_{L_2(\mathbb{R}^{nd})}$ for the vector and operator norms in $L_2(\mathbb{R}^{nd})$. The standard resolvent identity for Schrödinger operators combined with commutator estimates implies the following fact (cf. [14, Lemma 2.5.2]):

(GRI) *Geometric Resolvent Inequality*:

Let $\boldsymbol{\Lambda}_L(\mathbf{u})$, $\boldsymbol{\Lambda}_{\widetilde{L}}(\mathbf{u})$, $\mathbb{A}$ and $\mathbb{B}$ be as above. Then, $\forall\ E \in I \setminus \Big(\sigma\left(\mathbf{H}_{\boldsymbol{\Lambda}_L(\mathbf{u})}\right) \cup \sigma\Big(\mathbf{H}_{\boldsymbol{\Lambda}_{\tilde{L}}(\mathbf{u})}\Big)\Big)$, the operator norms satisfy

$$\|\mathbf{1}_{\mathbb{B}}\, \mathbf{G}^{\boldsymbol{\Lambda}_{\tilde{L}}(\mathbf{u})}(E)\, \mathbf{1}_{\mathbb{A}}\| \leq C^{(0)} \|\mathbf{1}_{\mathbb{B}}\, \mathbf{G}^{\boldsymbol{\Lambda}_{\tilde{L}}(\mathbf{u})}(E)\, \mathbf{1}_{\boldsymbol{\Lambda}^{\rm out}_L(\mathbf{u})}\| \times \|\mathbf{1}_{\boldsymbol{\Lambda}^{\rm out}}\, \mathbf{G}^{\boldsymbol{\Lambda}_L(\mathbf{u})}(E)\, \mathbf{1}_{\mathbb{A}}\|. \tag{2.2}$$

Here $C^{(0)} > 0$ is a 'geometric' constant: owing to the condition $4 \leq L < \tilde{L}$, this constant depends only on n (and is uniformly bounded for $1 \leq n \leq N$), but not on E. See [14, Lemma 2.5.4]. Later in this section, some other positive constants will appear, of a similar nature; we will denote them by $C^{(1)}$, $C^{(2)}$ and so on.

2.2. **Discretized Green functions.** Inequality (2.2) will enable us to use the function

$$\mathbf{B}_L(\mathbf{u}) \times \mathbf{B}_L(\mathbf{u}) \ni (\mathbf{v}, \mathbf{y}) \mapsto \| \,\mathbf{1}_{\mathbf{C}(\mathbf{v})}\, \mathbf{G}^{\boldsymbol{\Lambda}_L(\mathbf{u})}(E)\, \mathbf{1}_{\mathbf{C}(\mathbf{y})}\, \|$$

figuring in (1.27) as a discretization of original Green functions $\mathrm{G}^{\boldsymbol{\Lambda}_L(\mathbf{u})}(\mathbf{x}, \mathbf{x}'; E)$. Consequently, we will be able to apply a number of technical arguments developed earlier for multi-particle lattice Anderson models; see [7], [8], [9].

Let $L > 7$ and consider boxes $\boldsymbol{\Lambda}_L(\mathbf{u})$ and $\mathbf{B}_L(\mathbf{u})$. Further, pick a point $\mathbf{v} \in \mathbf{B}_L(\mathbf{u})$ and integer ℓ, with $3 < \ell < L-3$, such that $\boldsymbol{\Lambda}_\ell(\mathbf{v}) \subset \boldsymbol{\Lambda}_{L-3}(\mathbf{v})$. As above (see (1.25)), set

$$\boldsymbol{\Lambda}_L^{\rm out}(\mathbf{u}) = \boldsymbol{\Lambda}_L(\mathbf{u}) \setminus \boldsymbol{\Lambda}_{\tilde{L}-2}(\mathbf{u}) \text{ and } \boldsymbol{\Lambda}_\ell^{\rm out}(\mathbf{v}) = \boldsymbol{\Lambda}_\ell(\mathbf{v}) \setminus \boldsymbol{\Lambda}_{L-2}(\mathbf{v}), \tag{2.3a}$$

and

$$\mathbf{B}_L^{\rm out}(\mathbf{u}) = \boldsymbol{\Lambda}_L^{\rm out}(\mathbf{u}) \cap \mathbb{Z}^{nd}, \text{ and } \mathbf{B}_\ell^{\rm out}(\mathbf{v}) = \boldsymbol{\Lambda}_\ell^{\rm out}(\mathbf{v}) \cap \mathbb{Z}^{nd}. \tag{2.3b}$$

We have, evidently,

$$\boldsymbol{\Lambda}_L^{\rm out}(\mathbf{u}) \subset \bigcup_{\mathbf{w}\in\mathbf{B}_L^{\rm out}(\mathbf{u})} \mathbf{C}(\mathbf{w}) \text{ and } \boldsymbol{\Lambda}_\ell^{\rm out}(\mathbf{v}) \subset \bigcup_{\mathbf{w}\in\mathbf{B}_\ell^{\rm out}(\mathbf{v})} \mathbf{C}(\mathbf{w}).$$

Hence, for any $\mathbf{x} \in \mathbb{R}^{nd}$, the indicator functions obey

$$\mathbf{1}_{\boldsymbol{\Lambda}_L^{\rm out}(\mathbf{u})}(\mathbf{x}) \le \sum_{\mathbf{w}\in\mathbf{B}_L^{\rm out}(\mathbf{u})} \mathbf{1}_{\mathbf{C}(\mathbf{w})}(\mathbf{x}) \text{ and } \mathbf{1}_{\boldsymbol{\Lambda}_\ell^{\rm out}(\mathbf{v})}(\mathbf{x}) \le \sum_{\mathbf{w}\in\mathbf{B}_\ell^{\rm out}(\mathbf{v})} \mathbf{1}_{\mathbf{C}(\mathbf{w})}(\mathbf{x}). \tag{2.4}$$

Now the Eqn (2.2) implies, for $\mathbf{y} \in \mathbf{B}_L^{\rm out}(\mathbf{u})$ and $E \in I \setminus \Big(\sigma(\mathbf{H}_{\boldsymbol{\Lambda}_\ell(\mathbf{v})}) \cup \sigma(\mathbf{H}_{\boldsymbol{\Lambda}_L(\mathbf{u})}\Big)$,

$$\begin{aligned} &\|\mathbf{1}_{\mathbf{C}(\mathbf{v})}\,\mathbf{G}^{\boldsymbol{\Lambda}_L(\mathbf{u})}(E)\,\mathbf{1}_{\mathbf{C}(\mathbf{y})}\| \\ &\le C^{(0)} \sum_{\mathbf{w}\in\mathbf{B}_\ell^{\rm out}(\mathbf{v})} \|\mathbf{1}_{\mathbf{C}(\mathbf{v})}\,\mathbf{G}^{\boldsymbol{\Lambda}_\ell(\mathbf{v})}(E)\,\mathbf{1}_{\mathbf{C}(\mathbf{w})}\| \times \|\mathbf{1}_{\mathbf{C}(\mathbf{w})}\,\mathbf{G}^{\boldsymbol{\Lambda}_L(\mathbf{u})}(E)\,\mathbf{1}_{\mathbf{C}(\mathbf{y})}\|. \end{aligned} \tag{2.5}$$

Definition 2.1 (Discretized Green function)**.** Given boxes $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}_L(\mathbf{u})$ and $\mathbf{B} = \mathbf{B}_L(\mathbf{u})$, value $E \in \mathbb{R} \setminus \sigma(\mathbf{H}_{\boldsymbol{\Lambda}})$ and vectors $\mathbf{v}$, $\mathbf{w} \in \mathbf{B}$, we now denote

$$\mathrm{D}_{L,\mathbf{u}}(\mathbf{v},\mathbf{w};E) = \|\mathbf{1}_{\mathbf{C}(\mathbf{v})}\,\mathbf{G}^{\boldsymbol{\Lambda}}(E)\,\mathbf{1}_{\mathbf{C}(\mathbf{w})}\|. \tag{2.6}$$

We call function $\mathbf{B}\times\mathbf{B} \ni (\mathbf{v},\mathbf{w}) \mapsto \mathrm{D}_{L,\mathbf{u}}(\mathbf{v},\mathbf{w};E)$ the *discretized Green function* for $\mathbf{H}_{\boldsymbol{\Lambda}}$. The same definition is applicable for $\boldsymbol{\Lambda}_\ell(\mathbf{u})$ and $\mathbf{B}_\ell(\mathbf{u})$ yielding function $\mathbf{B}_\ell \times \mathbf{B}_\ell \ni (\mathbf{v},\mathbf{w}) \mapsto \mathrm{D}_{\ell,\mathbf{u}}(\mathbf{v},\mathbf{w};E)$.

It is worth to keep in mind that $\mathrm{D}_{L,\mathbf{u}}(\mathbf{v},\mathbf{w};E) = \mathrm{D}_{L,\mathbf{u}}(\mathbf{w},\mathbf{v};E) \ge 0$, $\mathbf{v},\mathbf{w} \in \mathbf{B}_L(\mathbf{u})$.

The bound in Eqn (2.5) now takes the following form:

(DGRI) *Discretized geometric resolvent inequality*: Given boxes $\boldsymbol{\Lambda}_\ell(\mathbf{v}) \subset \boldsymbol{\Lambda}_{L-3}(\mathbf{u})$, $\forall\, \mathbf{y} \in \mathbf{B}_L^{\rm out}(\mathbf{u})$ and $E \in I \setminus \Big(\sigma(\mathbf{H}_{\boldsymbol{\Lambda}_\ell(\mathbf{u})}) \cup \sigma(\mathbf{H}_{\boldsymbol{\Lambda}_L(\mathbf{u})})\Big)$,

$$\mathrm{D}_{L,\mathbf{u}}(\mathbf{v},\mathbf{y};E) \le C^{(0)} \sum_{\mathbf{w}\in\mathbf{B}_\ell^{\rm out}(\mathbf{v})} \mathrm{D}_{\ell,\mathbf{v}}(\mathbf{v},\mathbf{w};E)\,\mathrm{D}_{L,\mathbf{u}}(\mathbf{w},\mathbf{y};E). \tag{2.7}$$

Our task in the remaining part of the paper will be essentially reduced to the analysis of decay of functions $\mathrm{D}_{L_k,\mathbf{u}}(\mathbf{v},\mathbf{w};E)$ for $E \in \mathbb{R} \setminus \sigma(\mathbf{H}_{\boldsymbol{\Lambda}_{L_k}(\mathbf{u})})$, when vectors $\mathbf{v}$ and $\mathbf{w}$ are distant apart (viz., $\mathbf{v}$ is 'deeply' inside $\mathbf{B}_{L_k}(\mathbf{u})$ whereas $\mathbf{w}$ is near the boundary of $\mathbf{B}_{L_k}(\mathbf{u})$; see below).

Working with a lattice box $\mathbf{B} = \mathbf{B}_L(\mathbf{u}) \subset \mathbb{Z}^{nd}$, we will use the inner boundary $\partial^-\mathbf{B}$:

$$\partial^-\mathbf{B} := \{\mathbf{x} \in \mathbf{B} : \operatorname{dist}(\mathbf{x}, \mathbb{Z}^{nd} \setminus \mathbf{B}) = 1\}, \tag{2.8}$$

Similar notion can be introduced also for a general cellular set $\mathbb{A}$.

One of the key points in the proof of Theorem 1.13 is an exponential upper bound on discretized Green functions in finite boxes (cf. Eqn (3.1) in Lemma 3.1 below). This bound is obtained with the help of Lemma 3.7 using the notion of "lattice subharmonicity" introduced in the next section.

3. The scaling step inquality

The main result of this section is the bound (3.1) established in Lemma 3.1 below. It is in fact based on a construction alternative to [11, Lemma 4.2, Section 4] but serving the same purpose. A similar construction was used earlier in [9], in the framework of a multi-particle tight-binding Anderson model.

Lemma 3.1. *Given $n = 1, \dots, N$, $m > 0$, and a positive integer K, consider an n-particle box $\boldsymbol{\Lambda}_L(\mathbf{u})$. There exists a value $L^*_{\rm sc} = L^*_{\rm sc}(m, K)$ with the following property. Suppose that the conditions* (A)-(C) *are satisfied:*

(A) $L \geq L^*_{\rm sc}$.
(B) $\boldsymbol{\Lambda}_L(\mathbf{u})$ *is* E-CNR.
(C) *there exists a (possibly empty) family $\mathbf{A} = \{\mathbf{A}_i, 1 \leq i \leq J\}$ of disjoint annuli $\mathbf{A}_i = \boldsymbol{\Lambda}_{l_i+r_i}(u) \setminus \boldsymbol{\Lambda}_{l_i}(u)$ of total width $r_1 + \cdots + r_J \leq KL^{1/\alpha}$ such that any box $\boldsymbol{\Lambda}_\ell(\mathbf{v}) \subset \boldsymbol{\Lambda}_L(\mathbf{u}) \setminus \mathbf{A}$ is* NS.

Then box $\boldsymbol{\Lambda}_L(\mathbf{u})$ is NS*:*

$$\max_{\mathbf{y} \in \partial^- \mathbf{B}_L(\mathbf{u})} |\mathrm{G}^{\boldsymbol{\Lambda}_L(\mathbf{u})}(\mathbf{u}, \mathbf{y}; E)| \leq \mathrm{e}^{-\gamma(m,L,n)}. \tag{3.1}$$

The proof of Lemma 3.1 is completed at the end of the section; it is based on a number of auxiliary statements which occupy the rest of Section 3. Lemma 3.1 will be used in Section 6 with $n = N$ and $K = \kappa(N)$, where $\kappa(N) = N^n$ is the constant from Lemma 1.9.

3.1. **DGRI for NS boxes.** Suppose that a number $m > 0$ has been given, and consider an arbitrary point E from the bounded interval I. Consequently, we refer to (E, m)-NS and (E, m)-S boxes as NS- and S-boxes, assuming that E does not lie in the spectra of the corresponding operators.

The aim is is to derive, from the Eqn (2.7), an effective procedure of estimating the decay of the discretized Green functions $\mathrm{D}_{L,\mathbf{u}}(\mathbf{v}, \mathbf{w}; E)$ when vectors $\mathbf{v}$ and $\mathbf{w}$ are far from each other.

Given a positive integer $\ell < L$, assume that box $\boldsymbol{\Lambda}_L(\mathbf{u})$ does not contain an S-box $\mathbf{B}_\ell(\mathbf{v})$. Then Eqn (2.7) implies that for any site $\mathbf{y} \in \partial^- \mathbf{B}_L(\mathbf{u})$ and any box $\boldsymbol{\Lambda}_\ell(\mathbf{v}) \subset \boldsymbol{\Lambda}_L(\mathbf{u})$:

$$0 \leq \mathrm{D}_{L,\mathbf{u}}(\mathbf{v}, \mathbf{y}; E) \leq \mathrm{b}_1 \max_{\substack{\mathbf{w} \in \mathbf{B}_L(\mathbf{u}) \\ |\mathbf{w}-\mathbf{v}|=\ell}} \mathrm{D}_{L,\mathbf{u}}(\mathbf{w}, \mathbf{y}; E) \tag{3.2}$$

Here

$$\mathrm{b}_1 = C^{(1)} \mathrm{e}^{-m\ell} \ell^{Nd-1},$$

and $C^{(1)} = C^{(1)}(N)$ is another 'geometric' constant.

Definition 3.2 (E-complete non-resonance)**.** Set $\beta = 1/2$, $\alpha = 3/2$ (cf.1.24). Given $E \in I$ and $\mathbf{v} \in \mathbb{Z}^{nd}$, the n-particle box $\boldsymbol{\Lambda}_L(\mathbf{v})$ and the corresponding lattice box $\mathbf{B}_L(\mathbf{v})$ are called

(i) *E-nonresonant* (E-NR) if

$$\mathrm{dist}\big(E, \sigma(\mathbf{H}_{\boldsymbol{\Lambda}_L(\mathbf{v})})\big) \geq \mathrm{e}^{-L^\beta}, \tag{3.3}$$

and E-resonant (E-R) if the opposite inequality holds;

(ii) *E-completely non-resonant* (E-CNR) if $\boldsymbol{\Lambda}_L(\mathbf{v})$ is E-NR and does not contain any E-resonant box $\boldsymbol{\Lambda}^{(n)}_\ell(\mathbf{w})$ with $\ell \geq L^{1/\alpha}$.

3.2. **DGRI for non-resonant S-boxes.** Next, consider a situation where the box $\boldsymbol{\Lambda}_L(\mathbf{u})$ contains an (E,m)-S box $\boldsymbol{\Lambda}_\ell(\mathbf{v})$. Here $E \in I$, $m > 0$, $1 \le \ell < L$ and $\mathbf{v} \in \mathbf{B}_L(\mathbf{u})$.

Suppose that

(i) any box $\boldsymbol{\Lambda}_\ell(\mathbf{w}) \subset \boldsymbol{\Lambda}_L(\mathbf{u})$, with $\mathbf{w} \in \mathbf{B}_L(\mathbf{u})$ such that $\mathrm{dist}(\boldsymbol{\Lambda}_\ell(\mathbf{v}), \boldsymbol{\Lambda}_\ell(\mathbf{w})) = 1$, i.e., $|\mathbf{v}-\mathbf{w}| = 2\ell+1$, is NS;

(ii) all boxes $\boldsymbol{\Lambda}_{\ell'}(\mathbf{v}') \subset \boldsymbol{\Lambda}_L(\mathbf{u})$ with $\mathbf{v}' \in \mathbf{B}_L(\mathbf{u})$ and $\ell \le \ell' \le L$ are E-NR.

In this situation, Eqn (2.7) implies that, $\forall\ \mathbf{y} \in \mathbb{Z}^{nd} \cap \partial^- \boldsymbol{\Lambda}_L(\mathbf{u})$ and $\forall$ box $\boldsymbol{\Lambda}_\ell(\mathbf{v}) \subset \boldsymbol{\Lambda}_L(\mathbf{u})$,

$$\begin{aligned}\mathrm{D}_{L,\mathbf{u}}(\mathbf{v},\mathbf{y};E) \le C^{(0)} \mathrm{e}^{\ell^\beta} |\partial^- \boldsymbol{\Lambda}_\ell(\mathbf{v})| \max\ \Big[&\mathrm{D}_{L,\mathbf{u}}(\mathbf{w},\mathbf{y};E) : \\ &\mathbf{w} \in \mathbf{B}_L(\mathbf{u}), \boldsymbol{\Lambda}_\ell(\mathbf{w}) \subset \boldsymbol{\Lambda}_L(\mathbf{u}), |\mathbf{w}-\mathbf{v}| = 2\ell+1\Big].\end{aligned} \tag{3.4}$$

Applying Eqn (2.7) to all boxes $\boldsymbol{\Lambda}_\ell(\mathbf{w}) \subset \boldsymbol{\Lambda}_L(\mathbf{u})$ neighboring $\boldsymbol{\Lambda}_\ell(\mathbf{w})$, we get the bound

$$\begin{aligned}\mathrm{D}_{L,\mathbf{u}}(\mathbf{v},\mathbf{y};E) \le \mathrm{b}_2 \max\ \Big[&\mathrm{D}_{L,\mathbf{u}}(\mathbf{w},\mathbf{y};E) : \\ &\mathbf{w} \in \mathbf{B}_L(\mathbf{u}), \boldsymbol{\Lambda}_\ell(\mathbf{w}) \subset \boldsymbol{\Lambda}_L(\mathbf{u}), |\mathbf{w}-\mathbf{v}| = 2\ell+1\Big],\end{aligned} \tag{3.5}$$

with

$$\mathrm{b}_2 = C^{(2)} \mathrm{e}^{-m\ell} \mathrm{e}^{\ell^\beta} \ell^{d-1}, \tag{3.6}$$

where $C^{(0)} > 0$ is yet another 'geometric" constant.

Observe also that $\mathrm{b}_1 \le \mathrm{b}_2$, so that (3.2) implies a weaker inequality

$$\mathrm{D}_{L,\mathbf{u}}(\mathbf{u},\mathbf{y};E) \le \mathrm{b}_2 \max_{\mathbf{v} \in \partial^- \boldsymbol{\Lambda}_\ell(\mathbf{u})} \mathrm{D}_{L,\mathbf{u}}(\mathbf{v},\mathbf{y};E). \tag{3.7}$$

We see that the difference between the bounds (3.2) and (3.5) resides primarily in the form (and size) of the 'reference set' of points $\mathbf{w}$ under the sign of max.

3.3. **Multiple singular boxes.**

3.3.1. *Singular chain.*

Given a positive integer $\ell < L$ and an energy $E \in I$, suppose that $\boldsymbol{\Lambda}_L(\mathbf{u})$ contains some S-boxes of radius ℓ with centers in $\mathbf{B}_L(\mathbf{u})$. In order to be able to apply (3.5) to a given S-box $\boldsymbol{\Lambda}_\ell(\mathbf{v}^{(1)}) \subset \boldsymbol{\Lambda}_L(\mathbf{u})$, $\mathbf{v}^{(1)} \in \mathbf{B}_L(\mathbf{u})$, we would need all boxes of radius ℓ which neighbor $\boldsymbol{\Lambda}_\ell(\mathbf{v}^{(1)})$, lie in $\boldsymbol{\Lambda}_L(\mathbf{u})$ and are centered at a point in $\mathbf{B}_L(\mathbf{u})$ to be NS. However, one or more of these neighbors, say $\boldsymbol{\Lambda}_\ell(\mathbf{v}^{(2)})$, can be S. In such a case we pass to a larger box, $\boldsymbol{\Lambda}_{2\ell}(\mathbf{v}^{(1)}) \supset \boldsymbol{\Lambda}_\ell(\mathbf{v}^{(1)})$, and check for non-singularity of its neighbors $\boldsymbol{\Lambda}_\ell(\mathbf{v}^{(3)}) \subset \boldsymbol{\Lambda}_L(\mathbf{u}) \setminus \boldsymbol{\Lambda}_{2\ell}(\mathbf{v}^{(1)})$, with $\mathrm{dist}[\boldsymbol{\Lambda}_{2\ell}(\mathbf{v}^{(1)}), \boldsymbol{\Lambda}_\ell(\mathbf{v}^{(3)})] = 1$. Again, at least one of these boxes can be S. Then we pass to a larger box $\boldsymbol{\Lambda}_{3\ell}(\mathbf{v}^{(1)})$ and repeat the procedure. In the end we obtain a finite sequence of S-boxes

$$\boldsymbol{\Lambda}_\ell(\mathbf{v}^{(1)}), \dots, \boldsymbol{\Lambda}_\ell(\mathbf{v}^{(s)}) \subset \boldsymbol{\Lambda}_L(\mathbf{u}), \quad \text{where } s \ge 1,$$

with

$$\mathrm{dist}\big(\boldsymbol{\Lambda}_{(t-1)\ell-1}(\mathbf{v}^{(1)}), \boldsymbol{\Lambda}_\ell(\mathbf{v}^{(t)})\big) = 1, \quad 2 \le 2 \le s, \text{ when } s \ge 2.$$

We call such a sequence a singular chain, or, briefly, an S-chain, of length s.

It is not hard to see that if $\boldsymbol{\Lambda}_L(u)$ contains no S-chain of length $\ge K$, then for any point $\mathbf{y} \in \mathbf{B}_{L-2K\ell}(\mathbf{u})$ (i.e., not too close to the boundary of $\mathbf{B}_L(\mathbf{u})$) the following inequality holds true:

$$\begin{aligned}&\mathrm{D}_{L,\mathbf{u}}(\mathbf{v},\mathbf{y};E) \\ &\quad \le Q\ \max\ \big[\mathrm{D}_{L,\mathbf{u}}(\mathbf{w},\mathbf{y};E) :\ \mathbf{w} \in \mathbf{B}_L(\mathbf{u}), |\mathbf{w}-\mathbf{v}| = (A+1)\ell - 1\big].\end{aligned} \tag{3.8}$$

Here $A = A(\mathbf{v}) \le 2K$, and the factor $Q > 0$ is assessed by

$$Q \le C^{(3)}(2(A+1)\ell+1)^{nd-1}e^{-\gamma(m,\ell,n)}. \tag{3.9}$$

3.3.2. *Singular chains and separability.* Let $\kappa(n)$ be the value from Lemma 1.9. By Corollary 1.10, if we take $\kappa(n)+1$ disjoint annuli of width

$$B = B(n,\ell) = 2n\ell + 1 \tag{3.10}$$

with centre at $\mathbf{v}$:

$$\boldsymbol{A}_j(\mathbf{v}) = \boldsymbol{\Lambda}_{\ell+2jB}(\mathbf{v}) \setminus \boldsymbol{\Lambda}_{\ell+(2j-1)B}(\mathbf{v}), \quad 1 \le j \le \kappa(n)+1,$$

then at least one of them contains no box $\boldsymbol{\Lambda}_L(\mathbf{y})$ **not** separable from $\boldsymbol{\Lambda}_\ell(\mathbf{v})$.

Definition 3.3 (A bad box)**.** An n-particle box $\boldsymbol{\Lambda}_\ell(\mathbf{v})$ is called (E,m)-bad if it satisfies the following conditions:

- $\boldsymbol{\Lambda}_\ell(\mathbf{v})$ is (E,m)-singular;
- each annulus $\boldsymbol{A}_j(\mathbf{v})$, $1 \le j \le \kappa(n)+1$, contains an (E,m)-singular box $\boldsymbol{\Lambda}_\ell(\mathbf{w}_j)$.

The meaning of Definition 3.3 is that at least one of the (E,m)-singular boxes $\boldsymbol{\Lambda}_\ell(\mathbf{w})$ must be separable from $\boldsymbol{\Lambda}_\ell(\mathbf{v})$.

Definition 3.4 (The enveloping box)**.** Consider a finite, non-empty S-chain originating at $\mathbf{v}$ and assume that $\boldsymbol{\Lambda}_\ell(\mathbf{v})$ is not (E,m)-bad. The *enveloping box* for this S-chain associated is the smallest box $\boldsymbol{\Lambda}_{\widetilde{L}}(\mathbf{v})$ centered at $\mathbf{v}$ and containing this S-chain.

By construction of an enveloping box $\boldsymbol{\Lambda}_{\widetilde{L}}(\mathbf{v})$, any box of radius ℓ adjacent to its boundary $\partial\,\boldsymbol{\Lambda}_{\widetilde{L}}(\mathbf{v})$ must be NS. When we restrict ourselves to box $\boldsymbol{\Lambda}_L(\mathbf{u})$, we should always check if $\boldsymbol{\Lambda}_{\widetilde{L}}(\mathbf{v}) \subset \boldsymbol{\Lambda}_{L-2\ell-1}(\mathbf{u})$, i.e., box $\boldsymbol{\Lambda}_{\widetilde{L}}(\mathbf{v})$ lies at distance $\ge (2\ell+1)$ from the boundary $\partial^-\,\boldsymbol{\Lambda}_L(\mathbf{u})$, so that every box of radius ℓ neighboring $\boldsymbol{\Lambda}_{\widetilde{L}}(\mathbf{v})$ fits in $\boldsymbol{\Lambda}_L(\mathbf{u})$.

We finish subsection 3.3 with the following result concerning enveloping boxes.

Lemma 3.5. *Suppose that box $\boldsymbol{\Lambda}_L(\mathbf{u})$ contains no separable pair of (E,m)-singular boxes of radius ℓ. Then, for any point $\mathbf{v} \in \boldsymbol{\Lambda}_{L-(2\kappa(N)+1)\ell}(\mathbf{u})$, one of the following alternatives occurs:*

(1) $\boldsymbol{\Lambda}_\ell(\mathbf{v})$ is (E,m)-non-singular.
(2) There exists a box $\boldsymbol{\Lambda}_{\widetilde{L}}(\mathbf{v})$ of radius $\widetilde{L} \le (2\kappa(N)+1)\ell$ such that any box of radius ℓ adjacent to the boundary $\partial\,\boldsymbol{\Lambda}_{\widetilde{L}}(\mathbf{v})$ is (E,m)-non-singular.

Proof. In view of Corollary 1.10, given a point $\mathbf{v} \in \boldsymbol{\Lambda}_{L-(2\kappa(N)+1)\ell}(\mathbf{u})$, and a collection of at least $\kappa(n)+1$ disjoint annuli $\boldsymbol{A}_j(\mathbf{v}) = \boldsymbol{\Lambda}_{\ell+jR}(\mathbf{v}) \setminus \boldsymbol{\Lambda}_{\ell+(j-1)B}(\mathbf{v})$, at least one of these annuli contains only ℓ-boxes separable with $\boldsymbol{\Lambda}_\ell(\mathbf{v})$. □

3.4. Subharmonic functions in $\mathbf{B}_L(\mathbf{u})$.

3.4.1. *Formal definition.*

Definition 3.6 (Subharmonicity)**.** Fix constants $Q > 0$, $A > 1$ and integers $1 < \ell < L$, and let $\mathbf{S}$ be a subset in $\mathbf{B}_L(u)$. A nonnegative function $f\colon \mathbf{B}_L(\mathbf{u}) \to \mathbb{R}_+$ is called $(Q,A,\ell,\mathbf{S})$-*subharmonic* if it satisfies the following properties:

(i) $\forall$ point $\mathbf{x} \in \mathbf{B}_L(\mathbf{u}) \setminus \mathbf{S}$ with $\operatorname{dist}\big(\mathbf{x}, \partial^-\mathbf{B}_L(\mathbf{u})\big) \ge \ell$ we have

$$f(\mathbf{x}) \le Q \max\ \big[f(\mathbf{w}) :\ \mathbf{w} \in \mathbf{B}_L(\mathbf{u}), |\mathbf{w}-\mathbf{x}| = 2\ell+1\big]. \tag{3.11}$$

(ii) $\forall$ point $\mathbf{x}\in\mathbf{S}$ $\exists$ an integer $\rho(\mathbf{x})$ $\ell\le\rho(\mathbf{x})\le A\ell$, such that

$$f(\mathbf{x})\le Q\,\max\Big[f(\mathbf{w}):\ \mathbf{w}\in\mathbf{B}_L(\mathbf{u}),\ \rho(\mathbf{x})\le|\mathbf{w}-\mathbf{x}|\le\rho(\mathbf{x})+2\ell+1\Big] \tag{3.12}$$

Next, following [5, Lemma 4.3], we give a general bound for subharmonic functions.

Lemma 3.7. *Suppose that a function $f\colon\mathbf{B}_L(\mathbf{u})\to\mathbb{R}_+$ is $(Q,A,\ell,\mathbf{S})$-subharmonic, and that $\mathbf{S}$ can be covered by a collection of annuli with centre $\mathbf{u}$ of total width $W=W(\mathbf{S})$. Then*

$$f(\mathbf{u})\le Q^{(L-W-3\ell)/\ell}\max_{\mathbf{x}\in\mathbf{B}_L(\mathbf{u})}f(\mathbf{x}). \tag{3.13}$$

Proof. See [5, Proof of Lemma 4.3].

□

Lemma 3.8. *Consider a lattice box $\mathbf{B}_L(\mathbf{u})\subset\mathbb{Z}^{nd}$ and suppose that the following assumptions are satisfied:*

(1) $\mathbf{B}_L(\mathbf{u})$ is E-CNR;
(2) $\mathbf{B}_L(\mathbf{u})$ contains no (E,m)-bad box;
(3) all (E,m)-S boxes of radius ℓ inside $\mathbf{B}_L(\mathbf{u})$ can be covered by a set $\mathbf{S}$.

Then the function

$$f(\mathbf{x}):=\max_{\mathbf{y}\in\partial^-\mathbf{B}_L(\mathbf{u})}\mathrm{D}_{L,\mathbf{u}}(\mathbf{x},\mathbf{y};E) \tag{3.14}$$

is $(Q,\ell,\mathbf{S})$-subharmonic with

$$Q=C^{(4)}(d)(n\ell)^{d-1}\mathrm{e}^{\ell^\beta}\mathrm{e}^{-\gamma(m,\ell,n)}. \tag{3.15}$$

Proof. A straightforward application of Lemma 3.7.

Now suppose that any family of disjoint S-boxes

$$\mathbf{B}_\ell(\mathbf{v}^{(1)}),\ \mathbf{B}_\ell(\mathbf{v}^{(2)}),\dots,\mathbf{B}_\ell(\mathbf{v}^{(j)})\subset\mathbf{B}_L(\mathbf{u})\subset\mathbb{Z}^{nd}$$

corresponding to the cubes

$$\boldsymbol{\Lambda}_\ell(\mathbf{v}^{(1)}),\ \boldsymbol{\Lambda}_\ell(\mathbf{v}^{(2)}),\dots,\boldsymbol{\Lambda}_\ell(\mathbf{v}^{(j)})\subset\boldsymbol{\Lambda}_L(\mathbf{u})\subset\mathbb{R}^{nd}$$

contains at most J elements, for some fixed $J<\infty$. Then the function f defined in (3.14) is $(Q,\ell,\mathbf{S})$-subharmonic, with Q as in (3.15), and with some set $\mathbf{S}$ (in general, *not* unique) can be covered by a union $\mathbf{A}(\mathbf{S})$ of concentric annuli $\mathbf{A}_1$, …, $\mathbf{A}_j$:

$$\mathbf{A}_i=\mathbf{B}_{l_i+r_i}(\mathbf{u})\setminus\mathbf{B}_{l_i}(\mathbf{u}),\quad 1\le i\le j. \tag{3.16}$$

Here $0<l_1<l_1+r_1<l_2<\ldots<l_j+r_j<L$.

3.5. **Proof of Lemma 3.1.** Owing to Lemma 3.8, it suffices to apply Lemma 3.7 to functions $f:\ \mathbf{v}\mapsto\mathrm{D}_{L,\mathbf{u}}(\mathbf{v},\mathbf{y};E)$, $\mathbf{v}\in\mathbf{B}_L(\mathbf{u})$, with a fixed $\mathbf{y}\in\mathbf{B}_L(\mathbf{u})$.

4. The N-particle MSA induction scheme

In view of Theorem 1.12, our aim is to check property $\mathbf{DS}(m,p,k,I,n)$, i.e., (1.28), for $n=N$. As was mentioned before, it is done by means of a combined induction, in both k and N. Consequently, in some definitions below we refer to the particle number parameter n, whereas in other definitions - where we want to stress the passage from $N-1$ to N - we will use the capital letter.

The reader may assume from the start that the interval I is of ther form $[E^0, E^0+\eta]$.

We begin with the so-called Wegner-type bounds.

4.1. **Wegner-type bounds.** Given $n=1,\dots,N$, $q>0$ and $L_0\geq 2$, define two properties $\mathbf{W1}(n)(=\mathbf{W1}(n,q,L_0))$ and $\mathbf{W2}(n)(=\mathbf{W2}(n,q,L_0))$, for random n-particle Hamiltonians $\mathbf{H}^{(n)}_{\boldsymbol{\Lambda}}$ where $\boldsymbol{\Lambda}=\boldsymbol{\Lambda}^{(n)}_l(\mathbf{x})$ and $l\geq L_0$.

$\mathbf{W1}(n)$: For all $l\geq L_0$, for all $\mathbf{x}\in\mathbb{R}^{nd}$ and for all $E\in\mathbb{R}$,

$$\mathbb{P}\{\,\boldsymbol{\Lambda}^{(n)}_l(\mathbf{x}) \text{ is not } E\text{-CNR}\} < l^{-q}. \tag{4.1}$$

$\mathbf{W2}(n)$: Given a bounded interval $I\subset\mathbb{R}$, for all $l\geq L_0$ and any separable boxes $\boldsymbol{\Lambda}^{(n)}_\ell(\mathbf{x})$ and $\boldsymbol{\Lambda}^{(n)}_\ell(\mathbf{y})$,

$$\mathbb{P}\{\text{for some } E\in\mathbb{R},\text{ neither } \boldsymbol{\Lambda}^{(n)}_l(\mathbf{x}) \text{ nor } \boldsymbol{\Lambda}^{(n)}_l(\mathbf{y}) \text{ is } E\text{-CNR}\} < l^{-q}. \tag{4.2}$$

Theorem 4.1. *For any $q>0$ and a bounded interval $I\subset\mathbb{R}$, there exists $L^*_{\mathrm{W}}=L^*_{\mathrm{W}}(q,I)\in(0,+\infty)$ such that $\mathbf{W1}(n)$ and $\mathbf{W2}(n)$ hold true for all $n=1,\dots,N$ and $L_0\geq L^*_{\mathrm{W}}$.*

Proof. See [4].

4.2. **The initial step.** The initial step of the MSA induction consists in establishing properties $\mathbf{DS}(m,p,0,I,n)$ below, for $n=1,\dots,N$:

$\mathbf{DS}(m,p,0,I,n)$: $\forall$ pair of separable boxes $\boldsymbol{\Lambda}^{(n)}_{L_0}(\mathbf{u})$, $\boldsymbol{\Lambda}^{(n)}_{L_0}(\mathbf{v})$,

$$\mathbb{P}\big\{\text{ both } \boldsymbol{\Lambda}^{(n)}_{L_0}(\mathbf{u}) \text{ and } \boldsymbol{\Lambda}^{(n)}_{L_0}(\mathbf{v}) \text{ are } (E,m)\text{-S for some } E\in I\ \big\} < L_0^{-2p}. \tag{4.3}$$

We summarise it in Theorem 4.2:

Theorem 4.2. *Let $m>0$ and a positive integer L_0 be given. Then $\exists$ a value $\eta^*_0=\eta^*_0(m,L_0)>0$ with the following property.*

(i) *There exists a function*

$$(Nd,+\infty)\ni p\mapsto\eta_0(p)\in(0,\eta^*_0),\quad \textit{with } \eta_0(p)\searrow 0\ \ \textit{as}\ \ p\nearrow+\infty,$$

such that $\forall\, p>Nd$, Eqn (4.3) is satisfied $\forall\, n=1,\dots,N$ with $I=[E^0,E^0+\eta_0(p)]$.

(ii) *Equivalently, there exists a function*

$$(0,\eta^*_0)\ni\eta\mapsto p_0(\eta)>Nd,\quad \textit{with } p_0(\eta)\nearrow\infty\ \ \textit{as}\ \ \eta\searrow 0,$$

such that $\forall\,\eta\in(0,\eta^*_0)$, Eqn (4.3) is fulfilled $\forall\, n=1,\dots,N$, with $p=p_0(\eta)$ and $I=[E^0,E^0+\eta]$.

Proof. The assertion of Theorem 4.2 follows directly from [14, Theorems 2.2.3, 3.3.3] and is omitted from the paper. It is instructive to observe that the proofs in [14] do not rely upon a single- or multi-particle structure of the potential. $\square$

4.3. **The inductive step.** The inductive step of the MSA induction consists in deducing, given $k \geq 0$, property $\mathbf{DS}(m,p,k+1,I,N)$ from from properties $\mathbf{DS}(m,p,k,I,n)$ assumed for all $n = 1,\dots,N$ and properties $\mathbf{DS}(m,p,k+1,I,n)$ assumed for all $n = 1,\dots,N-1$. Let us summarise:

Theorem 4.3. *There exist values $L_+^* > 0$, $\eta_+^* > 0$, and two functions*

$$\eta \in (0,\eta_+^*) \mapsto p_+(\eta) > dN, \quad \textit{with} \;\; p_+(\eta) \xrightarrow[\eta\searrow 0]{} +\infty,$$

$$\eta \in (0,\eta_+^*) \mapsto m_+(\eta) > 0, \quad \textit{with} \;\; m_+(\eta) \xrightarrow[\eta\searrow 0]{} 0,$$

with the following property. $\forall$ *given $k \geq 0$, suppose that $0 < \eta < \eta_+^*$, $L_0 \geq L_+^*$ and*

- *property $\mathbf{DS}(m,p,k,I,n)$ holds with $I = [E^0, E^0+\eta]$, $m = m_+(\eta)$ and $p = p_+(\eta)$,* $\forall$ *$n = 1,\dots,N$ and*
- *property $\mathbf{DS}(m,p,k+1,I,n)$ holds with $I = [E^0, E^0+\eta]$, $m = m_+(\eta)$ and $p = p_+(\eta)$,* $\forall$ *$n = 1,\dots,N-1$.*

Then $\mathbf{DS}(m,p,k+1,I,N)$ also holds with $I = [E^0, E^0+\eta]$, $m = m_+(\eta)$ and $p = p_+(\eta)$.

The rest of the paper is devoted to the proof of Theorem 4.3. Observe that once this proof is completed, Theorem 1.13 and hence Theorem 1.5 will be established.

4.4. **Interactive boxes.** Recall: $\mathrm{r}_0 \in (0,+\infty)$ is the interaction radius (cf. (1.9)). Consider the following subset in $\mathbb{R}^{nd}$, $n = 1,\dots,N$:

$$\mathbb{D}^{(n)} = \big\{\mathbf{x} = (x_1,\dots,x_n) \in \mathbb{Z}^{nd} : \max_{1\leq j_1,\, j_2 \leq n} |x_{j_1} - x_{j_2}| \leq N\mathrm{r}_0\big\} \tag{4.4}$$

It is plain that, $\forall\, \mathbf{x} \in \mathbb{Z}^{nd} \setminus \mathbb{D}^{(n)}$,

$$\exists \text{ non-empty } \mathcal{J} \subset \{1,\dots,n\} \text{ such that } \min_{\substack{j_1 \in \mathcal{J} \\ j_2 \notin \mathcal{J}}} |x_{j_1} - x_{j_2}| > \mathrm{r}_0. \tag{4.5}$$

Definition 4.4 (Interactive boxes)**.** Let $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}_L^{(n)}(\mathbf{u})$ be an n-particle box. We say that

(i) $\boldsymbol{\Lambda}$ is *fully interactive* (FI) when $\boldsymbol{\Lambda} \cap \mathbb{D}^{(n)} \neq \varnothing$,
(ii) $\boldsymbol{\Lambda}$ is *partially interactive* (PI) when $\boldsymbol{\Lambda} \cap \mathbb{D}^{(n)} = \varnothing$.

The procedure of deducing property $\mathbf{DS}(m,p,k+1,I,N)$ from $\mathbf{DS}(m,p,k,I,N)$ and $\mathbf{DS}(m,p,k+1,I,n)$ with $n = 1,\dots,N-1$ is done separately for the following three types of pairs $\boldsymbol{\Lambda}_{L_{k+1}}^{(N)}(\mathbf{u})$, $\boldsymbol{\Lambda}_{L_{k+1}}^{(N)}(\mathbf{v})$ of separable boxes:

(I) Both $\boldsymbol{\Lambda}_{L_{k+1}}^{(N)}(\mathbf{u})$, $\boldsymbol{\Lambda}_{L_{k+1}}^{(N)}(\mathbf{v})$ are PI.
(II) Both $\boldsymbol{\Lambda}_{L_{k+1}}^{(N)}(\mathbf{u})$, $\boldsymbol{\Lambda}_{L_{k+1}}^{(N)}(\mathbf{v})$ are FI.
(III) One of $\boldsymbol{\Lambda}_{L_{k+1}}^{(N)}(\mathbf{u})$, $\boldsymbol{\Lambda}_{L_{k+1}}^{(N)}(\mathbf{v})$ is FI, while the other is PI.

These three cases are treated separately in Sections 5, 6 and 7, respectively. The end of proof of Theorem 4.3 is achieved at the end of Section 7.

5. Separable pairs of partially interactive singular boxes

In this section, we aim to derive property $\mathbf{DS}(m,p,k+1,I,N)$ in case (I), i.e., for a PI pair of separable boxes $\boldsymbol{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{u})$, $\boldsymbol{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{v})$. In this particular case we will be able to do this without referring to $\mathbf{DS}(m,p,k,I,n)$. However, we will use properties $\mathbf{DS}(m,p,k+1,I,n)$ for $n=1,\dots,N-1$. Cf. the statement of Theorem 5.6 below.

Let $\boldsymbol{\Lambda}=\boldsymbol{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{u})$ be a PI-box where $\mathbf{u}=(u_1,\dots,u_N)\in\mathbb{Z}^{Nd}$. Let $\mathcal{J}\subset\{1,\dots,N\}$ be a proper subset figuring in Eqn (4.5). Write $\mathbf{u}=(\mathbf{u}',\mathbf{u}'')$ where $\mathbf{u}'=\mathbf{u}_{\mathcal{J}}=(u_j)_{j\in\mathcal{J}}\in(\mathbb{Z}^d)^{\mathcal{J}}$ and $\mathbf{u}''=\mathbf{u}_{\mathcal{J}^c}=(u_j)_{j\notin\mathcal{J}}\in(\mathbb{Z}^d)^{\mathcal{J}^c}$ are the corresponding sub-configurations in $\mathbf{u}$. Let $n'=\#\mathcal{J}$ be the cardinality of $\mathcal{J}$ and $n''=N-n'$. We write $\boldsymbol{\Lambda}$ as the Cartesian product

$$\boldsymbol{\Lambda}=\boldsymbol{\Lambda}'\times\boldsymbol{\Lambda}'',\quad\text{where }\boldsymbol{\Lambda}'=\boldsymbol{\Lambda}^{(n')}_{L_{k+1}}(\mathbf{u}'),\ \boldsymbol{\Lambda}''\,\boldsymbol{\Lambda}^{(n'')}_{L_{k+1}}(\mathbf{u}'').$$

The Hamiltonian $\mathbf{H}^{(N)}_{\boldsymbol{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{u})}$ can be represented as

$$\mathbf{H}^{(N)}_{\boldsymbol{\Lambda}}=\mathbf{H}^{(n')}_{\boldsymbol{\Lambda}'}\otimes\mathbf{I}^{(n'')}+\mathbf{I}^{(n')}\otimes\mathbf{H}^{(n'')}_{\boldsymbol{\Lambda}''}. \tag{5.1}$$

Here $\mathbf{I}^{(n')}$ and $\mathbf{I}^{(n'')}$ are the identity operators on $L_2(\boldsymbol{\Lambda}')$ and $L_2(\boldsymbol{\Lambda}'')$, respectively. A similar decomposition can be written for each $\sigma(\mathbf{u})=(u_{\sigma(1)},\dots,u_{\sigma(N)})$, for any permutation σ of order N.

Definition 5.1 ((I,m)-partial tunneling)**.** In this definition we deal with $m>0$, $1\le n'\le N-1$, $k\ge 0$ and $\mathbf{u}'=(u_1,\dots,u_{n'})\in\mathbb{Z}^{nd'}$ and $\mathbf{u}=(u_1,\dots,u_N)\in\mathbb{Z}^{Nd}$ and a bounded interval $I\subset\mathbb{R}$ (eventually, of the form $I=[E^0,E^0+\eta]$).

(i) An n'-particle box $\boldsymbol{\Lambda}^{(n')}_{L_{k+1}}(\mathbf{u}')$ is (I,m)-*tunneling* (m-T) if there exists $E\in I$ and two separable n-particle boxes $\boldsymbol{\Lambda}^{(n')}_{L_k}(\mathbf{v}_j)\subset\boldsymbol{\Lambda}^{(n')}_{L_{k+1}}(\mathbf{u}')$, $j=1,2$ which are (E,m)-S.

(ii) An N-particle box $\boldsymbol{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{u})$ is (I,m,n',n'')-*partially tunelling* if
- $n'+n''=N$ and $n',n''\ge 1$,
- for $\mathbf{u}=(\mathbf{u}',\mathbf{u}'')$, $\mathbf{u}'=(u_1,\dots,u_{n'})$, $\mathbf{u}''=(u_{n'+1},\dots,u_N)$, we have

$$\boldsymbol{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{u})=\boldsymbol{\Lambda}^{(n')}_{L_{k+1}}(\mathbf{u}')\times\boldsymbol{\Lambda}^{(n'')}_{L_{k+1}}(\mathbf{u}''),$$

- either $\boldsymbol{\Lambda}^{(n')}_{L_{k+1}}(\mathbf{u}')$ or $\boldsymbol{\Lambda}^{(n')}_{L_{k+1}}(\mathbf{u}'')$ is (I,m)-tunneling.

(iii) An N-particle box $\boldsymbol{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{u})$ is (I,m)-*partially tunelling* ((I,m)-PT) if, for some permutation σ, $\boldsymbol{\Lambda}^{(N)}_{L_{k+1}}(\sigma(\mathbf{u}))$ is (I,m,n',n'')-partially tunelling for some $n',n''\ge 1$ with $n'+n''=N$. Otherwise, it is called (I,m)-*non partially tunelling* ((I,m)-NPT).

Lemma 5.2. *Consider an n-particle box of the form*

$$\boldsymbol{\Lambda}^{(n)}_{L_{k+1}}(\mathbf{u})=\boldsymbol{\Lambda}^{(n')}_{L_{k+1}}(\mathbf{u}')\times\boldsymbol{\Lambda}^{(n'')}_{L_{k+1}}(\mathbf{u}'')=\boldsymbol{\Lambda}'\times\boldsymbol{\Lambda}'',$$

where $\mathbf{u}=(\mathbf{u}',\mathbf{u}'')$*,* $\mathbf{u}'=(u_1,\dots,u_{n'})\in\mathbb{Z}^{nd'}$*,* $\mathbf{u}''=(u_{n'+1},\dots,u_n)\in\mathbb{Z}^{nd''}$*. Set:*

$$\boldsymbol{\Lambda}=\boldsymbol{\Lambda}^{(n)}_{L_{k+1}}(\mathbf{u}),\quad\boldsymbol{\Lambda}'=\boldsymbol{\Lambda}^{(n')}_{L_{k+1}}(\mathbf{u}'),\quad\boldsymbol{\Lambda}''=\boldsymbol{\Lambda}^{(n'')}_{L_{k+1}}(\mathbf{u}'').$$

(a) *Assume that* $\forall\ 1 \le j_1 \le n'$, $n'+1 \le j_2 \le n$, *we have*

$$|y_{j_1} - y_{j_2}| > \mathrm{r}_0,\ \forall\ \mathbf{y} = (y_1, \dots, y_N) \in \boldsymbol{\Lambda},$$

so that $\boldsymbol{\Lambda}$ *is* PI.

(b) *Assume also that box* $\boldsymbol{\Lambda}$ *is* (I,m)-NPT *for some* $m > 0$ *and* E-CNR *for some* $E \in I$ *where* $I \subset \mathbb{R}$ *is a bounded interval.*

Let $(E'_a, \boldsymbol{\Psi}'_a)$ *for* $a \ge 1$, *and* $(E''_b, \boldsymbol{\Psi}''_b)$ *for* $b \ge 1$ *be the eigenvalues and eigenvectors of* $\mathbf{H}^{(n')}_{\boldsymbol{\Lambda}'}$ *and* $\mathbf{H}^{(n'')}_{\boldsymbol{\Lambda}''}$, *respectively. Then, for* L_0 *large enough, the discretized Green functions obey:*

$$\begin{aligned} \max\left[\ \mathrm{D}_{L_{k+1},\mathbf{u}''}(\mathbf{u}'', \mathbf{v}''; E - E'_a)\ :\ a \ge 1,\ \mathbf{v}'' \in \partial^- \boldsymbol{\Lambda}''\ \right] &\le \mathrm{e}^{-\gamma(m, L_{k+1}, n)}\,, \\ \max\left[\ \mathrm{D}_{L_{k+1},\mathbf{u}'}(\mathbf{u}', \mathbf{v}'; E - E''_b)\ :\ b \ge 1,\ \mathbf{v}' \in \partial^- \boldsymbol{\Lambda}'\ \right] &\le \mathrm{e}^{-\gamma(m, L_{k+1}, n)}\,. \end{aligned} \tag{5.2}$$

This implies that box $\boldsymbol{\Lambda}$ *is* (E,m)-NS.

Proof. The proof is given in Section 8. □

Lemma 5.3. *Given* $m > 0$, $p > 0$, *a bounded interval* $I \subset \mathbb{R}$ *and* $n = 1, \dots, N-1$, *suppose that property* $\mathbf{DS(}m, p, k{+}1, I, n\mathbf{)}$ *holds for some* $k \ge 0$. *Then, for any* $\mathbf{u} \in \mathbb{Z}^{nd}$,

$$\mathbb{P}\big\{\boldsymbol{\Lambda}^{(n)}_{L_{k+1}}(\mathbf{u}) \text{ is } m\text{-PT}\big\} \le \frac{1}{2}|\boldsymbol{\Lambda}^{(n)}_{L_{k+1}}(\mathbf{u})|^2 \times L_k^{-2p} = \frac{1}{2}L_{k+1}^{-2p/\alpha+2d}. \tag{5.3}$$

Proof. Combine $\mathbf{DS(}m, p, k+1, I, n\mathbf{)}$ with a straightforward (albeit not sharp) upper bound $\frac{1}{2}|\boldsymbol{\Lambda}^{(n)}_{L_{k+1}}(\mathbf{u})|^2$ for the number of pairs $(\mathbf{y}_1,\, \mathbf{y}_2)$ of centers of boxes $\boldsymbol{\Lambda}^{(n)}_{L_k}(\mathbf{y}_j) \subset \boldsymbol{\Lambda}^{(n)}_{L_{k+1}}(\mathbf{u})$, $j = 1, 2$. □

Lemma 5.4. *Suppose* $m > 0$ *and a bounded interval* $I \subset \mathbb{R}$ *have been given. Let* $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{u})$, $\mathbf{u} = (u_1, \dots, u_N) \in \mathbb{Z}^{Nd}$, *be an* N*-particle* PI*-box of the form*

$$\boldsymbol{\Lambda} = \boldsymbol{\Lambda}' \times \boldsymbol{\Lambda}'' \quad \text{with } \boldsymbol{\Lambda}' = \boldsymbol{\Lambda}^{(n')}_{L_{k+1}}(\mathbf{u}'),\ \boldsymbol{\Lambda}'' = \boldsymbol{\Lambda}^{(n'')}_{L_{k+1}}(\mathbf{u}''),$$

where $n' + n'' = N$, $n', n'' \ge 1$, $\mathbf{u} = (\mathbf{u}', \mathbf{u}'')$, $\mathbf{u}' = (u_1, \dots, u_{n'})$, $\mathbf{u}'' = (u_{n'+1}, \dots, u_N)$. *Assume that* $\forall\ \mathbf{y} = (y_1, \dots, y_N) \in \boldsymbol{\Lambda}$,

$$\min_{\substack{1 \le i \le n' \\ n'+1 \le j \le N}} |y_i - y_j| > \mathrm{r}_0.$$

Then for any $p > 0$ *there exists* $\eta^*_{\mathrm{PT}} \in (0, +\infty)$ *such that the condition* $0 < \eta \le \eta^*_{\mathrm{PT}}$ *implies that*

$$\mathbb{P}\big\{\boldsymbol{\Lambda} \text{ is } m\text{-PT}\big\} \le \frac{1}{4}L_{k+1}^{-2p}. \tag{5.4}$$

Proof. By Definition 5.1, $\boldsymbol{\Lambda}$ is m–PT if and only if at least one of the boxes $\boldsymbol{\Lambda}'$ or $\boldsymbol{\Lambda}''$ is m-T. By Lemma 5.3, Eqn (5.3) holds for both $n = n'$ and $n = n''$. Since parameter $p_0(\eta) \to \infty$ as $\eta \to 0$ (see Theorem 4.2), this leads to the assertion of Lemma 5.4. □

Lemma 5.5. *Given* $L_0 \ge 1$, $m > 0$, $q > 0$, $p > \alpha(p_0(\eta) + d)$ *and a bounded interval* $I \subset \mathbb{R}$, *assume that*

- *the bound* (5.4) *holds true,*
- *for all* $n = 1, \dots, N-1$ *the bound* (5.3) *holds,*

- *L_0 is sufficiently large, so that for any $k \geq 0$ we have*

$$L_k^{-2p/\alpha+2d} \leq \frac{1}{4} L_k^{-2p_0(\eta)},$$

- *the bound (4.2) with $n = N$ (i.e., property $\mathbf{W2}(N)$) is satisfied.*

Then, for any integer $k \geq 0$ and for any pair of separable, PI *N-particle boxes $\boldsymbol{\Lambda}^{(N)}_{L_k}(\mathbf{u})$ and $\boldsymbol{\Lambda}^{(N)}_{L_k}(\mathbf{v})$,*

$$\mathbb{P}\{\boldsymbol{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{u}), \boldsymbol{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{v}) \text{ are } (E,m)\text{-S for some } E \in I\} \leq \frac{1}{2} L_{k+1}^{-2p_0(\eta)} + L_{k+1}^{-q}. \tag{5.5}$$

Proof. Set $\boldsymbol{\Lambda}(\mathbf{u}) = \boldsymbol{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{u})$ and $\boldsymbol{\Lambda}(\mathbf{v}) = \boldsymbol{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{v})$. Lemma 5.2 implies

$$\begin{aligned} &\mathbb{P}\{\boldsymbol{\Lambda}(\mathbf{u}) \text{ and } \boldsymbol{\Lambda}(\mathbf{v}) \text{ are } (E,m)\text{-S for some } E \in I\} \\ &\quad \leq \mathbb{P}\{\boldsymbol{\Lambda}(\mathbf{u}) \text{ or } \boldsymbol{\Lambda}(\mathbf{v}) \text{ is -PT}\} \\ &\qquad + \mathbb{P}\{\text{neither } \boldsymbol{\Lambda}(\mathbf{u}) \text{ nor } \boldsymbol{\Lambda}(\mathbf{v}) \text{ is } E\text{-CNR for some } E \in I\}. \end{aligned} \tag{5.6}$$

The assertion now follows from the assumptions of Lemma 5.5 and from the statement of Lemma 5.4. □

Theorem 5.6. *Given $p^* > Nd$, there exist $m^*_{\mathrm{PI}} > 0$, $\eta^*_{\mathrm{PI}} > 0$ and a positive $L^*_{\mathrm{PI}} < +\infty$ with the following property. Take $L_0 \geq L^*_{\mathrm{PI}}$. Then, $\forall\ k \geq 0$, $\mathbf{DS}(m, p, k+1, I, N)$ holds for all separable pairs of N-particle* PI*-boxes $\boldsymbol{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{u})$, $\boldsymbol{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{v})$ with $m = m^*_{\mathrm{PI}}$, $p = p^*$ and interval $I = [E^0, E^0 + \eta^*_{\mathrm{PI}}]$.*

Proof. The statement of Theorem 5.6 is an immediate corollary of Theorem 4.1 and Lemma 5.5. □

For future use, we also give

Lemma 5.7. *Given $0 < \eta < \min\ \left[\eta^*_0, \eta^*_{\mathrm{PI}}\right]$, $L_0 \geq 1$, $q > 0$, $p \geq 2p_0(\eta) + 2d$ and a bounded interval $I \subset \mathbb{R}$, assume that*

- *the bound (5.4) holds true,*
- *for all $n = 1, \dots, N-1$ the bound (5.3) holds,*
- *L_0 is sufficiently large, so that for any $k \geq 0$ we have*

$$L_k^{-2p/\alpha+2d} \leq \frac{1}{4} L_k^{-2p_0(\eta)},$$

- *the bound (4.1) with $n = N$ (i.e., property $\mathbf{W1}(N)$) is satisfied.*

Let $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{u})$ be an N-particle box. Let $\nu_{\mathrm{PI}}(\boldsymbol{\Lambda}; E)$ be the (random) maximal number of (E,m)-S, pairwise separable PI*-boxes $\boldsymbol{\Lambda}^{(N)}_{L_k}(\mathbf{y}) \subset \boldsymbol{\Lambda}$. Then the following inequality takes place:*

$$\mathbb{P}\big\{\nu_{\mathrm{PI}}(\boldsymbol{\Lambda}; E) \geq 2 \text{ for some } E \in I\big\} \leq \frac{1}{2} L_k^{2d\alpha} \Big(L_k^{-2p_0(\eta)} + L_k^{-q}\Big). \tag{5.7}$$

Proof. If $\nu_{\mathrm{PI}} \geq 2$, then there exist (at least) two singular boxes $\boldsymbol{\Lambda}_{L_k}(\mathbf{x}), \boldsymbol{\Lambda}_{L_k}(\mathbf{y})$. The number of possible pairs $(\mathbf{x}, \mathbf{y})$ is bounded by $\frac{1}{2} L_{k+1}^{2d}$, while for a given pair $\boldsymbol{\Lambda}_{L_k}(\mathbf{x})$, $\boldsymbol{\Lambda}_{L_k}(\mathbf{y})$ Lemma 5.5 applies. This leads to the assertion of Lemma 5.7. □

6. Separable pairs of fully interactive singular boxes

The main outcome in case (II) is Theorem 6.5 at the end of this section. Recall, the definition of an FI-box was related to $\mathrm{r}_0 \in (0, +\infty)$, the radius of interaction (cf. (1.9)). Further, the definition of a separable pair of boxes $\boldsymbol{\Lambda}_L(\mathbf{u})$ and $\boldsymbol{\Lambda}_L(\mathbf{v})$ was related to the constant R, the diameter of support of the bump functions and included the condition

$$\mathrm{dist}\ (\boldsymbol{\Lambda}_L(\mathbf{u}), \boldsymbol{\Lambda}_L(\mathbf{v})) > 2N(L+R)$$

(see Definition 1.8). Before we proceed further, let us state a geometric assertion:

Lemma 6.1. *Let $L > \mathrm{r}_0$ be an integer. Let $\boldsymbol{\Lambda}_L(\mathbf{u}')$ and $\boldsymbol{\Lambda}_L(\mathbf{u}'')$ be two separable N-particle* FI*-boxes, where $\mathbf{u}' = (u'_1, \dots, u'_N)$, $\mathbf{u}'' = (u''_1, \dots, u''_N)$. Then*

$$\Pi\ \boldsymbol{\Lambda}_{L+R}(\mathbf{u}') \cap \Pi\ \boldsymbol{\Lambda}_{L+R}(\mathbf{u}'') = \varnothing. \tag{6.1}$$

Proof. If $\boldsymbol{\Lambda}_L(\mathbf{u}')$ is FI, then there exists a permutation σ of order N such that, for all $j = 1, \dots, N-1$,

$$|u'_j - u'_{j+1}| \le \mathrm{r}_0.$$

Otherwise, the set $\{u'_j\}_{1 \le j \le N} \subset \mathbb{Z}^d$ could be decomposed into two or more non-interacting subsets. Therefore,

$$\mathrm{diam}\{u'_j\}_{1\le j\le n} \le (N-1)\mathrm{r}_0; \text{ similarly, } \mathrm{diam}\{u''_j\}_{1\le j\le n} \le (N-1)\mathrm{r}_0.$$

Further, suppose that for some $i, j \in \{1, \dots, n\}$, we have

$$\Pi_i\ \boldsymbol{\Lambda}_{L+R}(\mathbf{u}') \cap \Pi_j\ \boldsymbol{\Lambda}_{L+R}(\mathbf{u}'') \ne \varnothing.$$

Then $|u'_i - u''_j| \le 2(L+R)$, and, therefore, for any $k = 1, \dots, n$

$$\begin{aligned} |u'_k - u''_k| &\le |u'_k - u'_i| + |u'_i - u''_j| + |u''_j - u''_k| \\ &\le (N-1)\mathrm{r}_0 + 2(L+R) + (N-1)\mathrm{r}_0 \\ &\le 2N(L+R). \end{aligned}$$

This is incompatible with the inequality $\mathrm{dist}(\boldsymbol{\Lambda}_L(\mathbf{u}'), \boldsymbol{\Lambda}_L(\mathbf{u}'')) > 2nN(L+R)$, since in the latter case there must exist some k such that $|u'_k - u''_k| > 2(L+R)$. ☐

Lemma 6.1 is used in the proof of Lemma 6.2 which, in turn, is a part of the proof of Lemma 6.3, instrumental in establishing Theorem 6.5.

Let an interval $I \subset \mathbb{R}$ and a number $m > 0$ be given. Consider the following assertion which is a particular case of $\mathbf{DS(}m, p, k, I, N\mathbf{)}$ (cf. Eqn 1.28):

$\mathbf{FIS(}k, p, N\mathbf{)}$: For any pair of separable N-particle FI-boxes $\boldsymbol{\Lambda}^{(N)}_{L_k}(\mathbf{u})$ and $\boldsymbol{\Lambda}^{(N)}_{L_k}(\mathbf{v})$

$$\mathbb{P}\big\{\boldsymbol{\Lambda}^{(N)}_{L_k}(\mathbf{u}) \text{ and } \boldsymbol{\Lambda}^{(N)}_{L_k}(\mathbf{v}) \text{ are } (E,m)\text{-S for some } E \in I\big\} \le L_k^{-2p}. \tag{6.2}$$

Lemma 6.2. *Let $k \ge 0$ be given. Assume that property $\mathbf{FIS(}k, p, N\mathbf{)}$ holds true. Let $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{u})$ be an N-particle box. Denote by $\nu_{\mathrm{FI}}(\boldsymbol{\Lambda}; E)$ the (random) maximal number of (E,m)-S, pairwise separable* FI*-boxes $\boldsymbol{\Lambda}^{(N)}_{L_k}(\mathbf{y}^{(j)}) \subset \boldsymbol{\Lambda}$. Then, for any $\ell \ge 1$,*

$$\mathbb{P}\Big\{\nu_{\mathrm{FI}}(\boldsymbol{\Lambda}; E) \ge 2\ell \textit{ for some } E \in I\Big\} \le L_k^{2\ell(1+d\alpha)??} \cdot L_k^{-2\ell p}. \tag{6.3}$$

Proof. Suppose there exist FI-boxes $\boldsymbol{\Lambda}^{(N)}_{L_k}(\mathbf{y}^{(j)}) \subset \boldsymbol{\Lambda}$, $j = 1, \dots, 2\ell$, such that any two of them are separable. By virtue of Lemma 6.1, it is readily seen that the pairs of operators $\Big(\mathbf{H}_{\boldsymbol{\Lambda}_{L_k}(\mathbf{y}^{(2i-1)})}, \mathbf{H}_{\boldsymbol{\Lambda}_{L_k}(\mathbf{y}^{(2i)})}\Big)$, $i = 1, \dots, \ell$, form an independent family. [It is also true that, within a given pair, operators $\mathbf{H}_{\boldsymbol{\Lambda}_{L_k}(\mathbf{y}^{(2i-1)})}(\omega)$ and $\mathbf{H}_{\boldsymbol{\Lambda}_{L_k}(\mathbf{y}^{(2i)})}(\omega)$ are mutually independent.]

Thus, any collection of events $\mathcal{A}_1, \dots, \mathcal{A}_\ell$ related to these pairs also forms an independent family. Now, for $i = 1, \dots, \ell$, set

$$\mathcal{A}_i = \Big\{ \boldsymbol{\Lambda}_{L_k}(\mathbf{y}^{(2i-1)}) \text{ and } \boldsymbol{\Lambda}_{L_k}(\mathbf{y}^{(2i)}) \text{ are } (E,m)\text{-S for some } E \in I \Big\}. \tag{6.4}$$

Then, owing to property $\mathbf{FIS}(k,p,N)$ (see (6.2)), $\forall\ i = 1, \dots, \ell$,

$$\mathbb{P}\left\{\mathcal{A}_i\right\} \le L_k^{-2p}, \tag{6.5}$$

and by virtue of independence of events $\mathcal{A}_1, \dots, \mathcal{A}_\ell$, we obtain that

$$\mathbb{P}\left\{\bigcap_{i=1}^{\ell} \mathcal{A}_i\right\} = \prod_{i=1}^{\ell} \mathbb{P}\left\{\mathcal{A}_i\right\} \le \left(L_k^{-2p}\right)^\ell. \tag{6.6}$$

To complete the proof, note that the total number of different families of 2ℓ boxes $\boldsymbol{\Lambda}_{L_k}^{(N)} \subset \boldsymbol{\Lambda}_{L_{k+1}}^{(N)}(\mathbf{u})$ with required properties is bounded from above by

$$\frac{1}{(2\ell)!}\left[2(L_k + \mathrm{r}_0 + 1)L_{k+1}^d\right]^{2\ell} \le \frac{1}{(2\ell)!}\left(3L_kL_{k+1}^d\right)^{2\ell} \le L_k^{2\ell(1+d\alpha)}.$$

In fact, their centres must lie at distance $\le L_k + \mathrm{r}_0$ from the set $\mathbb{D}^{(N)} \cap \mathbf{B}_{L_{k+1}}^{(N)}(\mathbf{u})$. This yields the assertion of Lemma 6.2.

□

Lemma 6.3. *Given $k \ge 0$, let $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}_{L_{k+1}}^{(N)}(\mathbf{u})$ be an N-particle box. Consider an interval I of the form $I = [E^0, E^0 + \eta]$ and assume that the conditions of Lemmas 5.7 and 6.2 are fufilled. Given $E \in I$, denote by $\nu_{\mathrm{S}}(\boldsymbol{\Lambda}; E)$ the (random) maximal number of (E,m)-S, pairwise separable boxes $\boldsymbol{\Lambda}_{L_k}^{(N)}(\mathbf{u}^{(j)}) \subset \boldsymbol{\Lambda}$. Let $\kappa(N)$ be the constant from Lemma 1.9. Then, $\forall\ \ell \ge 1$,*

$$\begin{aligned}&\mathbb{P}\{\nu_{\mathrm{S}}(\boldsymbol{\Lambda}; E) \ge 2\ell + \kappa(N) + 1 \textit{ for some } E \in I\}\\ &\qquad \le L_k^{4d\alpha} \cdot L_k^{-2p_0(\eta)} + L_k^{2\ell(1+d\alpha)??} \cdot L_k^{-2\ell p}.\end{aligned} \tag{6.7}$$

Proof. Suppose that $\nu_{\mathrm{S}}(\boldsymbol{\Lambda}; E) \ge 2\ell + \kappa(N) + 1$. Let $\nu_{\mathrm{PI}}(\boldsymbol{\Lambda}; E)$ be as in Lemma 5.7 and $\nu_{\mathrm{FI}}(\boldsymbol{\Lambda}; E)$ as in Lemma 6.2. Obviously,

$$\nu_{\mathrm{S}}(\boldsymbol{\Lambda}; E) \le \nu_{\mathrm{PI}}(\boldsymbol{\Lambda}; E) + \nu_{\mathrm{FI}}(\boldsymbol{\Lambda}; E).$$

Then either $\nu_{\mathrm{PI}}(\boldsymbol{\Lambda}; E) \ge \kappa(N) + 1$ or $\nu_{\mathrm{FI}}(\boldsymbol{\Lambda}; E) \ge 2\ell$. Therefore,

$$\begin{aligned}&\mathbb{P}\{\nu_{\mathrm{S}}(\boldsymbol{\Lambda}; E) \ge 2\ell + \kappa(N) + 1 \text{ for some } E \in I\}\\ &\qquad \le \mathbb{P}\{\nu_{\mathrm{PI}}(\boldsymbol{\Lambda}; E) \ge \kappa(N) + 1 \text{ for some } E \in I\}\\ &\qquad\quad + \mathbb{P}\{\nu_{\mathrm{FI}}(\boldsymbol{\Lambda}; E) \ge 2\ell \text{ for some } E \in I\}\\ &\qquad \le L_k^{4d\alpha} \cdot L_k^{-2p_0(\eta)} + L_k^{2\ell(1+d\alpha)} \cdot L_k^{-2\ell p},\end{aligned}$$

by virtue of (5.7) and (6.3). □

An elementary calculation gives rise to the following

Corollary 6.4. *Under assumptions of Lemma 6.3, with $\ell \ge 2$, $p_0(\eta)$ and p large enough and for L_0 large enough, we have, for any integer $k \ge 0$,*

$$\mathbb{P}\{\nu_{\mathrm{S}}(\boldsymbol{\Lambda}; E) \ge 2\ell + 2 \textit{ for some } E \in I\} \le L_{k+1}^{-2p-1}. \tag{6.8}$$

Now, if two N-particle boxes $\boldsymbol{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{u}')$ and $\boldsymbol{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{u}'')$ are separable, then property $\mathbf{W2}(N)$ (i.e., Eqn (4.2) with $n = N$) implies the bound

$$\begin{aligned}&\mathbb{P}\{\text{for any } E \in I, \text{ either } \boldsymbol{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{u}') \text{ or } \boldsymbol{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{u}'') \text{ is } E\text{-CNR}\}\\ &\qquad \geq 1 - L_{k+1}^{-(q\alpha^{-1}-2\alpha)} > 1 - L_{k+1}^{-(q'(N)-4)}.\end{aligned} \tag{6.9}$$

Here $q' := q/\alpha$.

The main result of this section is the following

Theorem 6.5. *For any $p^* > Nd$ large enough, there exist $m^*_{\mathrm{FI}} > 0$, $\eta^*_{\mathrm{FI}} > 0$ and $L^*_{\mathrm{FI}} \in (0, +\infty)$ such that the following property holds true. Given $L_0 \geq L^*_{\mathrm{FI}}$ and $k \geq 0$, assume that property $\mathbf{FIS}(k, p, N)$ holds with $m = m^*_{\mathrm{FI}}$, $p = p^*$ and interval $I = [E^0, E^0 + \eta^*_{\mathrm{FI}}]$. Then property $\mathbf{FIS}(k + 1, p, N)$ also holds, again with $m = m^*_{\mathrm{FI}}$, $p = p^*$ and interval $I = [E^0, E^0 + \eta^*_{\mathrm{FI}}]$.*

Proof. Let $m > 0$, $\mathbf{u}, \mathbf{v} \in \mathbb{Z}^{Nd}$ and assume that $\boldsymbol{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{u})$ and $\boldsymbol{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{v})$ are separable FI-boxes. With an interval I of the form $[E^0, E^0 + \eta]$, consider the following two events:

$$\begin{aligned}\mathcal{B} &= \Big\{\boldsymbol{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{u}) \text{ and } \boldsymbol{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{v}) \text{ are } (E, m)\text{-S for some } E \in I\Big\},\\ \mathcal{D} &= \Big\{\text{for some } E \in I, \text{ neither } \boldsymbol{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{u}) \text{ nor } \boldsymbol{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{v}) \text{ is } E\text{-CNR}\Big\}.\end{aligned}$$

The argument that follows assumes that parameters m, η, p and L_0 are adjusted in the way specified in the conditions of Theorem 6.5. Owing to property $\mathbf{W2}(N)$ (cf. Eqn (4.2), with $n = N$), we have:

$$\mathbb{P}(\mathcal{D}) < L_{k+1}^{-(q'-4)}, \text{ where } q' := \frac{q}{\alpha}. \tag{6.10}$$

Moreover, $\mathbb{P}(\mathcal{B}) \leq \mathbb{P}(\mathcal{D}) + \mathbb{P}(\mathcal{B} \cap \mathcal{D}^{\mathrm{c}})$. So, it suffices to estimate the probability $\mathbb{P}(\mathcal{B} \cap \mathcal{D}^{\mathrm{c}})$. Within the event $\mathcal{B} \cap \mathcal{D}^{\mathrm{c}}$, for any $E \in I$, either $\boldsymbol{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{u})$ or $\boldsymbol{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{v})$ must be E-CNR. Without loss of generality, assume that for some $E \in I$, $\boldsymbol{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{u})$ is E-CNR and (E, m)-S. By Lemma 3.1, if L_0 (and, therefore, any L_k) is sufficiently large, for such value of E, $\nu_{\mathrm{S}}(\boldsymbol{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{u}); E) \geq K + 1$, with K as in Lemma 3.1. Now let $K = \kappa(N)$, where $\kappa(N)$ is the contant from Lemma 1.9. We see that

$$\mathcal{B} \cap \mathcal{D}^{\mathrm{c}} \subset \big\{\nu_{\mathrm{S}}(\boldsymbol{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{u}); E) \geq \kappa(N) + 1 \text{ for some } E \in I\big\}$$

and, therefore, by Lemma 6.3 and Corollary 6.4,

$$\mathbb{P}(\mathcal{B} \cap \mathcal{D}^{\mathrm{c}}) \leq \mathbb{P}\{\exists\, E \in I \,|\, \nu_{\mathrm{S}}(\boldsymbol{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{u}); E) \geq \kappa(N) + 1\} \leq L_k^{-2p}. \tag{6.11}$$

□

7. Mixed separable pairs of singular boxes

It remains to derive property $\mathbf{DS}(m, p, k+1, I, N)$ in case (III), i.e., for mixed pairs of N-particle boxes (where one is FI and the other PI).

A natural counterpart of Theorem 6.5 for mixed pairs of boxes is the following

Theorem 7.1. *For any $p^* > Nd$ large enough, there exist $m^*_{\mathrm{MI}} > 0$, $\eta^*_{\mathrm{MI}} > 0$ and $L^*_{\mathrm{MI}} \in (0, +\infty)$ guaranteing the following property. Given $L_0 \geq L^*_{\mathrm{FI}}$ and $k \geq 0$, assume that property $\mathbf{DS}(m, p, k, I, N)$ holds, with $m = m^*_{\mathrm{MI}}$, $p = p^*$ and interval $I = [E^0, E^0 + \eta^*_{\mathrm{MI}}]$,*

- *for any pair of separable* PI-*boxes $\boldsymbol{\Lambda}^{(N)}_{L_k}(\mathbf{x})$, $\boldsymbol{\Lambda}^{(N)}_{L_k}(\mathbf{y})$, $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^{Nd}$,*

- *for any pair of separable* FI*-boxes* $\boldsymbol{\Lambda}^{(N)}_{L_k}(\tilde{\mathbf{x}})$, $\boldsymbol{\Lambda}^{(N)}_{L_k}(\tilde{\mathbf{y}})$, $\mathbf{x},\mathbf{y}\in\mathbb{Z}^{Nd}$.

Then property $\mathbf{DS}(m,p,k+1,I,N)$ *holds for mixed pairs of separable boxes* $\boldsymbol{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{u})$ *and* $\boldsymbol{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{v})$.

In other words, if $\boldsymbol{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{u})$, $\boldsymbol{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{v})$ *is a mixed pair of separable boxes then, for* $p=p^*$, $m=m^*_{\rm MI}$ *and* $I=[E^0,E^0+\eta^*_{\rm MI}]$,

$$\mathbb{P}\big\{\boldsymbol{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{x}) \text{ and } \boldsymbol{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{y}) \text{ are } (E,m)\text{-}S \text{ for some } E\in I\big\}\le L_{k+1}^{-2p}. \tag{7.1}$$

Proof. Assume that $\boldsymbol{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{u})$, $\boldsymbol{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{v})$ is separable pair where box $\boldsymbol{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{u})$ is FI and $\boldsymbol{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{v})$ PI. Consider the following three events:

$$\begin{aligned}
&\mathcal{B}=\big\{\exists\, E\in I:\ \boldsymbol{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{u}) \text{ and } \boldsymbol{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{v}) \text{ are } (E,m)\text{-S}\big\},\\
&\mathcal{T}=\big\{\boldsymbol{\Lambda}_{L_{k+1}}(\mathbf{v}) \text{ is } (I,m)\text{-PT}\big\},\\
&\mathcal{D}=\big\{\exists\, E\in I:\ \text{ neither } \boldsymbol{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{u}) \text{ nor } \boldsymbol{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{v}) \text{ is } E\text{-CNR}\big\}.
\end{aligned}$$

By virtue of (3.4??),

$$\mathbb{P}(\mathcal{T})\le\frac{1}{4}L_{k+1}^{-2p},$$

and by Theorem 4.1,

$$\mathbb{P}(\mathcal{D})\le L_{k+1}^{-q+2}.$$

Further,

$$\mathbb{P}(\mathcal{B})\le\mathbb{P}(\mathcal{T})+\mathbb{P}(\mathcal{B}\cap\mathcal{T}^{\rm c})\le\frac{1}{4}L_{k+1}^{-2p}+\mathbb{P}(\mathcal{B}\cap\mathcal{T}^{\rm c}).$$

Thus, we have

$$\mathbb{P}(\mathcal{B}\cap\mathcal{T}^{\rm c})\le\mathbb{P}(\mathcal{D})+\mathbb{P}(\mathcal{B}\cap\mathcal{T}^{\rm c}\cap\mathcal{D}^{\rm c})\le L_{k+1}^{-q+2}+\mathbb{P}(\mathcal{B}\cap\mathcal{T}^{\rm c}\cap\mathcal{D}^{\rm c}).$$

Next, within the event $\mathcal{B}\cap\mathcal{T}^{\rm c}\cap\mathcal{D}^{\rm c}$, either $\boldsymbol{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{u})$ or $\boldsymbol{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{v})$ is E-CNR. It must be the FI-box $\boldsymbol{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{u})$. Indeed, by Lemma 5.2, had box $\boldsymbol{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{v})$ been both E-CNR and (I,m)-NPT, it would have been (E,m)-NS, which is not allowed within the event $\mathcal{B}$. Thus, the box $\boldsymbol{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{u})$ must be E-CNR, but (E,m)-S. Hence,

$$\mathcal{B}\cap\mathcal{T}^{\rm c}\cap\mathcal{D}^{\rm c}\subset\big\{\exists\, E\in I:\boldsymbol{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{u}) \text{ is } (E,m)\text{-S and } E\text{-CNR}\big\}.$$

However, applying Lemma 3.7, we see that

$$\begin{aligned}
&\{\exists\, E\in I:\ \boldsymbol{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{u}) \text{ is } (E,m)\text{-S and } E\text{-CNR}\}\\
&\qquad\subset\{\exists\, E\in I:\ \nu_{\rm S}(\boldsymbol{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{u});E)\ge 2\ell+\kappa(N)+1\}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbb{P}(\mathcal{B}\cap\mathcal{T}^{\rm c}\cap\mathcal{D}^{\rm c})&\le\mathbb{P}\{\exists\, E\in I:\ \nu_{\rm S}(\boldsymbol{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{u});E)\ge 2\ell+\kappa(N)+1\}\\
&\le 2L_{k+1}^{-1}\,L_{k+1}^{-2p}. \qquad (7.2)
\end{aligned}$$

Finally, we get, with $q':=q/\alpha$,

$$\begin{aligned}
\mathbb{P}(\mathcal{B})&\le\mathbb{P}(\mathcal{B}\cap\mathcal{T})+\mathbb{P}(\mathcal{D})+\mathbb{P}(\mathcal{B}\cap\mathcal{T}^{\rm c}\cap\mathcal{D}^{\rm c})\\
&\le\frac{1}{2}L_{k+1}^{-2p}+L_{k+1}^{-q'(N)+4}+2L_{k+1}^{-1}L_{k+1}^{-2p}\le L_{k+1}^{-2p}, \qquad (7.3)
\end{aligned}$$

for sufficiently large L_0, if we can guarantee, by taking $\eta > 0$ small enough, that $q'(N) > 2p + 5$. This completes the proof of Theorem 7.1. $\square$

Therefore, Theorem 4.3 is also proven.

8. Appendix. Proof of Lemmas 1.9 and 5.2

Proof of Lemma 1.9. Given a positive integer L a non-empty set $\mathcal{J} \subset \{1, \dots, n\}$ and an n-particle vector $\mathbf{y} = (y_1, \dots, y_n) \in \mathbb{R}^{nd}$ we say that the set of positions $\{y_j\}_{j\in\mathcal{J}}$, forms an $(L+R)$-clump if the union

$$\bigcup_{j\in\mathcal{J}} \Lambda_R(x_j) \subset \mathbb{R}^d \tag{8.1}$$

yields a connected set. Next, consider two n-particle vectors $\mathbf{x}$ and $\mathbf{y}$ and proceed as follows.

1) Decompose the vector $\mathbf{y}$ into maximal L-clumps $\Gamma_1, \dots, \Gamma_M$ (of diameter $\leq 2nL$ each), with the total number M of clumps being $\leq n$.
2) To each position y_i there corresponds precisely one clump, Γ_j where $j = j(i) \in \{1, \dots, M\}$.
3) Suppose that there exists $j \in \{1, \dots, M\}$ such that $\Gamma_j \cap \Pi\, \mathbf{\Lambda}^{(n)}_{L+R}(\mathbf{x}) = \varnothing$. Then boxes $\mathbf{\Lambda}^{(n)}_L(\mathbf{y})$ and $\mathbf{\Lambda}^{(n)}_L(\mathbf{x})$ are separable.
4) Suppose 3) is wrong; the aim is to deduce from the negation of 3) a necessary condition on possible locations of vector $\mathbf{y}$ and assess the number of possible choices. Indeed our hypothesis reads:

$$\Gamma_j \cap \Pi\, \mathbf{\Lambda}^{(N)}_{L+R}(\mathbf{x}) \neq \varnothing \quad \text{for some } j = 1, \dots, M. \tag{8.2}$$

Therefore,

$$\left.\begin{array}{r} \forall\, j = 1, \dots, M,\ \exists\, i \text{ such that} \\ |y_j - x_i| \leq \operatorname{dist}(y_j, \partial\Gamma_j) + \operatorname{dist}(\partial\Gamma_j, x_i) \\ \leq [2n(L+R) - (L+R)] + L + R = 2n(L+R) \end{array}\right\} \implies \begin{cases} \forall\, j = 1, \dots, M, \\ y_j \in \Pi\, \mathbf{\Lambda}^{(n)}_{A(L+R)}(\mathbf{x}) \\ \text{with } A \leq 2n. \end{cases}$$

We see that if a configuration $\mathbf{y}$ is not separable from $\mathbf{x}$, then every position y_j must belong to one of the boxes $\Pi_i\, \mathbf{\Lambda}^{(n)}_{AL}(\mathbf{x}) = \Lambda_{AL}(x_i) \subset \mathbb{Z}^d$. The total number of such boxes is $\leq n$. There are at most n^n choices of the boxes $\Lambda_{AL}(x_i)$ for n positions $y_1, \dots, y_n$; so we set $\kappa(n) = n^n$. For any given choice among $\leq \kappa(n)$ possibilities, the point $\mathbf{y} = (y_1, \dots, y_N n)$ must belong to the Cartesian product of n boxes of size AL, i.e., to an (nd)-dimensional box of size AL. The first assertion of Lemma 1.9 now follows.
5) Next, consider a particular case where

$$\mathbf{\Lambda}^{(n)}_{L+R}(\mathbf{y}) \bigcap \mathbf{\Lambda}^{(n)}_{|\mathbf{x}|+L+R}(\mathbf{0}) = \varnothing.$$

Then there exists at least one value of $i \in \{1, \dots, N\}$ such that

$$\Pi_i\, \mathbf{\Lambda}^{(n)}_{L+R}(\mathbf{y}) \bigcap \Pi_i\, \mathbf{\Lambda}^{(n)}_{|\mathbf{x}|+L+R}(\mathbf{0}) = \varnothing. \tag{8.3}$$

However, by symmetry of the centered box $\mathbf{\Lambda}^{(n)}_{|\mathbf{x}|+L+R}(\mathbf{0})$ with respect to permutation of the coordinates, the projections $\Pi_i\, \mathbf{\Lambda}^{(n)}_{|\mathbf{x}|+L+R}(\mathbf{0})$ are identical:

$$\Pi\, \mathbf{\Lambda}^{(n)}_{|\mathbf{x}|+L+R}(\mathbf{0}) = \Pi_i\, \mathbf{\Lambda}^{(n)}_{|\mathbf{x}|+L+R}(\mathbf{0}),\ i = 1, \dots, N.$$

This implies separability of boxes $\boldsymbol{\Lambda}_L^{(n)}(\mathbf{y})$ and $\boldsymbol{\Lambda}_{|\mathbf{x}|+L}^{(n)}(\mathbf{0})$.

This completes the proof of Lemma 1.9. □

We now pass to the proof of Lemma 5.2. Recall that we consider an n-particle box of the form

$$\boldsymbol{\Lambda} = \boldsymbol{\Lambda}_{L_{k+1}}^{(n)}(\mathbf{u}), \quad \boldsymbol{\Lambda}' = \boldsymbol{\Lambda}_{L_{k+1}}^{(n')}(\mathbf{u}'), \quad \boldsymbol{\Lambda}'' = \boldsymbol{\Lambda}_{L_{k+1}}^{(n'')}(\mathbf{u}''),$$

with $\mathbf{u} = (\mathbf{u}', \mathbf{u}'') \in \mathbb{Z}^{nd}$, $\mathbf{u}' \in \mathbb{Z}^{n'd}$, $\mathbf{u}'' \in \mathbb{Z}^{n''d}$. The corresponding Hamiltonian $\mathbf{H}_{\boldsymbol{\Lambda}}^{(n)}$ has the following form:

$$\mathbf{H}_{\boldsymbol{\Lambda}}^{(n)} = \mathbf{H}_{\boldsymbol{\Lambda}'}^{(n')} \otimes \mathbf{1}_{\boldsymbol{\Lambda}''}^{(n'')} + \mathbf{1}_{\boldsymbol{\Lambda}'}^{(n')} \otimes \mathbf{H}_{\boldsymbol{\Lambda}''}^{(n'')}.$$

Further, let $\{\boldsymbol{\Psi}'_a, a \geq 1\}$ be normalized eigenfunctions of $\mathbf{H}_{\boldsymbol{\Lambda}'}^{(n')}$ and $\{E'_a, a \geq 1\}$ the corresponding eigenvalues. Correspondingly, we denote by $\{\boldsymbol{\Psi}''_b, b \geq 1\}$ and $\{E''_b, b \geq 1\}$ (normalized) eigenfunctions and eigenvalues of operator $\mathbf{H}_{\boldsymbol{\Lambda}'}^{(n')}$. Then the normalized eigenfunctions and respective eigenvalues of $\mathbf{H}_{\boldsymbol{\Lambda}}^{(n)}$ can be chosen in the form

$$\boldsymbol{\Psi}_{a,b} := \boldsymbol{\Psi}'_a \otimes \boldsymbol{\Psi}''_b, \ E_{a,b} = E'_a + E''_b, \ a, b \geq 1.$$

We assume that $E'_{a+1} \geq E'_a$, $E''_{b+1} \geq E''_b$, $a, b \geq 1$.

Proof of Lemma 5.2. By hypothesis, $\boldsymbol{\Lambda}$ is E-CNR. Therefore, $\forall\, a, b \geq 1$

$$\begin{aligned} e^{-L_{k+1}^{\beta}} &< |E - E_{a,b}| = |E - (E'_a + E''_b)| \\ &= |(E - E'_a) - E''_b| = |(E - E''_b) - E'_a| \end{aligned}$$

Therefore,

- for all E'_a, the n''-particle box $\boldsymbol{\Lambda}''$ is $(E - E'_a)$-NR;
- for all E''_b, the n'-particle box $\boldsymbol{\Lambda}'$ is $(E - E''_b)$-NR.

By the assumption of (I, m)-NPT, for all $E \in I$ the box $\boldsymbol{\Lambda}''$ should not contain two separable $(E - E'_a, m)$-S sub-boxes of radius L_k. Therefore, the assumptions of Lemma 3.1 hold true, and we deduce that the box $\boldsymbol{\Lambda}''$ is $(E - E'_a)$-NS, yielding the required upper bound for $\boldsymbol{\Lambda}''$.

The box $\boldsymbol{\Lambda}'$ is also (I, m)-NPT, by the hypothesis of the lemma, so the same argument applies to $\boldsymbol{\Lambda}'$.

Let us now prove that box $\boldsymbol{\Lambda}$ is (E, m)-NS. If $\mathbf{v} = (\mathbf{v}', \mathbf{v}'') \in \partial\, \boldsymbol{\Lambda}_{L_{k+1}}^{(n)}(\mathbf{u})$, then either $|\mathbf{u}' - \mathbf{v}'| = L_{k+1}$, or $|\mathbf{u}'' - \mathbf{v}''| = L_{k+1}$. First, consider the case where $|\mathbf{u}' - \mathbf{v}'| = L_{k+1}$. In this case we can write the Green functions as

$$\begin{aligned} \mathbf{G}^{\boldsymbol{\Lambda}}(\mathbf{u}, \mathbf{v}; E) &= \sum_a \boldsymbol{\Psi}'_a(\mathbf{u}')\boldsymbol{\Psi}'_a(\mathbf{v}') \sum_b \frac{\boldsymbol{\Psi}''_b(\mathbf{u}'')\boldsymbol{\Psi}''_b(\mathbf{v}'')}{(E - E'_a) - E''_b} \\ &= \sum_a \boldsymbol{\Psi}'_a(\mathbf{u}')\boldsymbol{\Psi}'_a(\mathbf{v}')\, \mathrm{G}^{\boldsymbol{\Lambda}''}(\mathbf{u}'', \mathbf{v}''; E - E'_a). \end{aligned} \tag{8.4}$$

For the resolvent operators we have the representation:

$$\mathbf{G}^{\boldsymbol{\Lambda}}(E) = \sum_a \mathbf{P}'_{\boldsymbol{\Psi}'_a} \otimes \mathrm{G}^{\boldsymbol{\Lambda}''}(E - E'_a).$$

Here $\mathbf{P}_{\boldsymbol{\Psi}'_a}$ is the orthogonal projection on the (normalized) eigenfunction $\boldsymbol{\Psi}'_a$. Naturally, $\|\mathbf{P}_{\boldsymbol{\Psi}'_a}\| = 1$. Recall that we aim to bound the norm

$$\begin{aligned}
\| \mathbf{1}_{\mathbf{C}(\mathbf{u})} \mathbf{G}^{\boldsymbol{\Lambda}}(E) \mathbf{1}_{\mathbf{C}(\mathbf{v})} \| &= \| \mathbf{1}_{\mathbf{C}'(\mathbf{u}')} \otimes \mathbf{1}_{\mathbf{C}''(\mathbf{u}'')} \ \mathbf{G}^{\boldsymbol{\Lambda}}(E) \mathbf{1}_{\mathbf{C}'(\mathbf{v}')} \mathbf{1}_{\mathbf{C}''(\mathbf{v}'')} \| \\
&= \| \mathbf{1}_{\mathbf{C}'(\mathbf{u}')} \otimes \mathbf{1}_{\mathbf{C}''(\mathbf{u}'')} \ \mathbf{G}^{\boldsymbol{\Lambda}}(E) \mathbf{1}_{\mathbf{C}'(\mathbf{v}')} \mathbf{1}_{\mathbf{C}''(\mathbf{v}'')} \| \\
\le \sum_a \| \left(\mathbf{1}_{\mathbf{C}'(\mathbf{u}')} \mathbf{P}_{\boldsymbol{\Psi}'_a} \mathbf{1}_{\mathbf{C}'(\mathbf{v}')} \right) &\otimes \left(\mathbf{1}_{\mathbf{C}''(\mathbf{u}'')} \otimes \mathrm{G}^{\boldsymbol{\Lambda}''}(E - E'_a) \mathbf{1}_{\mathbf{C}''(\mathbf{v}'')} \right) \| \\
&\le \sum_a \| \mathbf{1}_{\mathbf{C}''(\mathbf{u}'')} \otimes \mathrm{G}^{\boldsymbol{\Lambda}''}(E - E'_a) \mathbf{1}_{\mathbf{C}''(\mathbf{v}'')} \|.
\end{aligned} \tag{8.5}$$

Since the interaction potential U and the external random potential $V(x;\omega)$ are non-negative, the eigenvalues E'_a satisfy $E'_a \ge E'_a(0) > 0$ where $E'_a(0)$ are the eigenvalues of the operator $-\frac{1}{2}\boldsymbol{\Delta}$ in the n'-particle box $\boldsymbol{\Lambda}'$, by min-max principle. Eigenvalues $E'_a(0) \nearrow \infty$ as $a \to \infty$, and their growth rate is controlled by the Weyl formula.

This allows to perform an effective cut-off of the series in the RHS of (8.5). Namely, let $\delta > 1$ be fixed, then the following quantity is well-defined:

$$A(\delta, \eta) := \max\{a \ge 1 \,|: \ \eta - E'_a \ge -\delta\}.$$

Moreover,

$$\begin{gathered}
A(\delta, \eta) \le C_{\mathrm{Weyl}}(\delta, n'd) |\mathbf{B}^{(n')}_{L_{k+1}}(\mathbf{u}')| \\
E - E'_a \le E - E'_a(0).
\end{gathered}$$

Here

$$C_{Weyl}(n'd, \delta) = \frac{\delta^{n'd/2}}{\Gamma\left(1 + \frac{n'd}{2}\right)(4\pi)^{n'd/2}} < \delta^{n'd/2};$$

so that we can use a more explicit upper bound $A(\delta, \eta) \le \delta^{n'd/2} |\mathbf{B}^{(n')}_{L_{k+1}}(\mathbf{u}')|$.

Further, for any $a \ge A(\delta, \eta)$ we have $E - E'_a \le -\delta < 0$, so that the distance between the point $E - E'_a$ and the spectrum of operator $\mathbf{H}_{\boldsymbol{\Lambda}''}$ is $> \delta$. Then, by virtue of the Combes-Thomas estimate [1][10],

$$|| \mathbf{G}^{\boldsymbol{\Lambda}''}(E - E'_a)|| < e^{-c|E - E'_a|\,|u-v|} < e^{-c\delta|u-v|}$$

Now we chose δ large enough, thus making the exponent $c\delta$ arbitrarily large. Taking into account the rate of growth of $E'_a \ge E'_a(0)$, we can write

$$\begin{aligned}
&\sum_{a > A_0(\delta,\eta)} \| \mathbf{1}_{\mathbf{C}''(\mathbf{u}'')} \otimes \mathrm{G}^{\boldsymbol{\Lambda}''}(E - E'_a) \mathbf{1}_{\mathbf{C}''(\mathbf{v}'')} \| \\
&\qquad \le \sum_{a > A_0(\delta,\eta)} e^{-c|E - E'_a| \|\mathbf{u}'' - \mathbf{v}''\|} \le C_1 e^{-C_2 \delta L_{k+1}}
\end{aligned}$$

Next, we have to estimate the norm of a finite sum

$$\begin{aligned}
&\sum_{a=1}^{A_0(\delta,\eta)} \| \mathbf{1}_{\mathbf{C}''(\mathbf{u}'')} \otimes \mathrm{G}^{\boldsymbol{\Lambda}''}(E - E'_a) \mathbf{1}_{\mathbf{C}''(\mathbf{v}'')} \| \\
&\le A_0(\delta, \eta) \max_{1 \le a \le A_0(\delta,\eta)} \| \mathbf{1}_{\mathbf{C}''(\mathbf{u}'')} \otimes \mathrm{G}^{\boldsymbol{\Lambda}''}(E - E'_a) \mathbf{1}_{\mathbf{C}''(\mathbf{v}'')} \| \\
&\le A_0(\delta, \eta) e^{-\gamma(m, L_{k+1},)} \le \delta^{n'd/2} |\mathbf{B}_{L_{k+1}}(\mathbf{u}')| e^{-\gamma(m, L_{k+1})}
\end{aligned}$$

[1] For small values of the distance δ, the decay exponent in the Green functions is of order of $\sqrt{\delta}$, cf. [3]. Here the original Combes–Thomas bound is stronger for large $|E - E'_a|$.

where we used again that $\|\mathbf{u}'' - \mathbf{v}''\| = L_{k+1}$. Combining the two bounds, we obtain

$$\begin{aligned}
&\sum_{a=1}^{\infty} \| \, \mathbf{1}_{\mathbf{C}''(\mathbf{u}'')} \otimes \mathrm{G}^{\boldsymbol{\Lambda}''}(E - E'_a) \, \mathbf{1}_{\mathbf{C}''(\mathbf{v}'')} \, \| \\
&\leq \delta^{n'd/2} |\mathbf{B}_{L_{k+1}}(\mathbf{u}')| e^{-\gamma(m, L_{k+1})} + C_1 e^{-C_2 \delta L_{k+1}} \\
&\leq 2\delta^{n'd/2} |\mathbf{B}_{L_{k+1}}(\mathbf{u}')| e^{-\gamma(m, L_{k+1})},
\end{aligned}$$

for sufficiently large L_0 (hence, large L_{k+1}). Now recall that the function γ has the form

$$\gamma(m, L, n) \; (= \gamma_N(m, L, n)) \;\; = mL \left(1 + L^{-1/4}\right)^{N-n}, \; 1 \leq n \leq N,$$

so that, for $n' \leq n - 1$, we have

$$\gamma(m, L_{k+1}, n') \geq \gamma(m, L_{k+1}, N - 1) = mL_{k+1} \left(1 + L_{k+1}^{-1/4}\right)^{(N-n)+1}$$

and

$$\begin{aligned}
&- \ln \left(2\delta^{n'd/2} |\mathbf{B}_{L_{k+1}}(\mathbf{u}')| e^{-\gamma(m, L_{k+1}, n-1)}\right) \\
&= mL_{k+1} \left(1 + L_{k+1}^{-1/4}\right)^{N-n+1} - C \ln L_{k+1} \\
&= L_{k+1} \left(1 + L_{k+1}^{-1/4}\right)^{N-n+1} \left(m - CL_{k+1}^{-1} \ln L_{k+1}\right) \\
&\geq L_{k+1} \left(1 + L_{k+1}^{-1/4}\right)^{N-n} m \, \left(1 + L_{k+1}^{-1/4}\right) \left(1 - L_{k+1}^{-1/2}\right)
\end{aligned}$$

(provided that $L_{k+1}^{1/2} \geq Cm^{-1} \ln L_{k+1}$, which is true for sufficiently large L_0)

$$\geq L_{k+1} \left(1 + L_{k+1}^{-1/4}\right)^{N-n} m,$$

provided that $L_0 > 16$.

Finally, note that in the case where $|\mathbf{u}'' - \mathbf{v}''| = L_{k+1}$, we can use the representation

$$\mathrm{G}^{\boldsymbol{\Lambda}}(\mathbf{u}, \mathbf{v}; E) = \sum_b \boldsymbol{\Psi}''_b(\mathbf{u}'') \boldsymbol{\Psi}''_b(\mathbf{v}'') \, \mathrm{G}^{\boldsymbol{\Lambda}'}(\mathbf{u}', \mathbf{v}'; E - E''_b) \tag{8.6}$$

and repeat the previous argument. □

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