Anderson localization in a multi-particle continuous model with an alloy-type external potential

A. Boutet de Monvel[1], **V. Chulaevsky**[2], **P. Stollmann**[3], **Y. Suhov**[4]

[1]Institut de Mathématiques de Jussieu
Université Paris 7
175 rue du Chevaleret, 75013 Paris, France
E-mail: aboutet@math.jussieu.fr

[2]Département de Mathématiques
Université de Reims, Moulin de la Housse, B.P. 1039,
51687 Reims Cedex 2, France
E-mail: victor.tchoulaevski@univ-reims.fr

[3] Fakultät für Mathematik
Technische Universität Chemnitz
09107 Chemnitz, Germany
E-mail: peter.stollmann@mathematik.tu-chemnitz.de

[4] Statistical Laboratory, DPMMS
University of Cambridge, Wilberforce Road,
Cambridge CB3 0WB, UK
E-mail: Y.M.Suhov@statslab.cam.ac.uk

Abstract: We establish the exponential localization in a multi-particle Anderson model in a Euclidean space $\mathbb{R}^d$, $d \geq 1$, in presence of a non-trivial short-range interaction and a random external potential of an alloy type. Specifically, we prove all eigenfunctions with eigenvalues near the lower edge of the spectrum decay exponentially in $\mathcal{L}_2$-norm.

1. Introduction. The N-particle Hamiltonian in the continuum

1A. The model. This paper considers an N-particle Anderson model in $\mathbb{R}^d$ with interaction. The Hamiltonian $\mathbf{H}\left(=\mathbf{H}^{(N)}(\omega)\right)$ is a random Schrödinger operator of the form

$$\mathbf{H} = -\frac{1}{2}\mathbf{\Delta} + \mathbf{U}(\mathbf{x}) + \mathbf{V}(\omega;\mathbf{x}) \tag{1.1}$$

acting in $\mathcal{L}_2(\mathbb{R}^d\times\ldots\times\mathbb{R}^d) \simeq \mathcal{L}_2(\mathbb{R}^d)^{\otimes N}$. This means that we consider N quantum particles, each living in $\mathbb{R}^d$, in the following fashion. The joint position vector is $\mathbf{x} = (x_1,\ldots,x_N) \in \mathbb{R}^d \times \ldots \times \mathbb{R}^d$, where component $x_j = \left(\mathrm{x}_j^{(1)},\ldots,\mathrm{x}_j^{(d)}\right) \in \mathbb{R}^d$ represents the coordinates of the j's particle, $j = 1,\ldots,N$. Next,

$$-\frac{1}{2}\mathbf{\Delta} = -\frac{1}{2}\sum_{1\leq j\leq N} \Delta_j$$

is the standard kinetic energy operator obtained by adding up the kinetic energies $-\dfrac{1}{2}\Delta_j$ of individual particles and assuming that the particles are of identical masses. In the case of different masses, $-\frac{1}{2}\mathbf{\Delta}$ would have been replaced by the sum $-\frac{1}{2}\sum_{1\le j\le N}\frac{1}{m_j}\Delta_j$, without changing the analysis involved. As usually, Δ_j stands for the Laplacian $\sum_{i=1}^{d}\frac{\partial^2}{\partial {\rm x}_j^{(i)\,2}}$ in $\mathbb{R}^d$.

The interaction energy operator is denoted by $\mathbf{U}(\mathbf{x})$: it is the operator of multiplication by a function $\mathbf{x}\in\mathbb{R}^d\times\ldots\times\mathbb{R}^d\mapsto U(\mathbf{x})$, the inter-particle potential (which can also incorporate a deterministic external potential). Finally, the term $\mathbf{V}(\omega;\mathbf{x})$ represents the operator of multiplication by a function

$$\mathbf{x}\mapsto V(\omega;x_1)+\ldots+V(\omega;x_N),\quad \mathbf{x}=(x_1,\ldots,x_N)\in\mathbb{R}^d\times\ldots\times\mathbb{R}^d, \qquad (1.2)$$

where $x\in\mathbb{R}^d\mapsto V(x;\omega)$, $x\in\mathbb{R}^d$, is the random external field potential. Assumptions on $U(\mathbf{x})$ and $V(\omega;x)$ are discussed below; in essence, U is required to be a sum of short-range inter-particle potentials while V is assumed to be of an alloy type.

We will analyse spectral properties of operator $\mathbf{H}$ by using the so-called Multi-Scale Analysis (MSA) method, more precisely, its 'continuous-space' version. Our goal here is two-fold.

- First, we show that of the continuous-space version of the MSA can be reduced, in a certain way, to its discrete counterpart, for an auxiliary lattice problem, and the corresponding argument works equally well for systems with several particles. The MSA is known to be a powerful and versatile method successfully applied to a number of spectral problems in random media. It was originally developed for a single-particle lattice (tight-binding) Anderson models (cf. [FS83, FMSS85], [DK89]) and later adapted to problems in a Euclidean space. See [BCH97, CH94, DS01, HM84, KSS98A, KSS98B, K95]; the monograph [St01] contains more complete references up to the year 2000. Notable later developments, still for one-particle systems, are [GK01], [BK05] (solving the notorious problem of localization for a Bernoulli–Anderson model in $\mathbb{R}^d$), and [AGKW08], where the MSA was adapted to a large class of singular distributions of the external random potential.

However, the existing continuous adaptations are technically more involved than the original lattice version of the MSA. This leads to greater complexity of the continuous localization analysis, particularly when one attempts to treat the case of more than one particle. In contrast, the reduction to an auxiliary lattice problem presented in this paper works in a fairly general fashion. This reduction is encapsulated in the so-called Geometric Resolvent Inequalities (GRI) and allows a direct application of lattice techniques (and some additional general facts) in a ready-made form which is technically much less involved.

In addition, we simplify an important ingredient of the many-particle lattice MSA, following the strategy outlined in [C08]. As a result, a relatively straightforward strategy of the proof of Anderson localization emerges, applicable for both discrete and continuous multi-particle models.

- Second, we combine in this paper the above-mentgioned reduction techniques with the method from [CS09A] (and in part from [CS09B]), where a multi-particle tight-binding Anderson localization has been proved, for large disorders. As a result, we prove here Anderson localization near the lower edge of the spectrum for an N-particle model in a Euclidean space $\mathbb{R}^d$, $d \geq 1$, with an alloy-type external random potential and a short-range interaction between particles. See Theorem 1.1 below. An essential ingredient here are the so-called Wegner-type bounds obtained for alloy-type systems in [BCSS08A], [BCSS08B].

1B. Basic concepts. Throughout this paper, we work with cubes with the edges parallel to the co-ordinate axes in the Euclidean spaces $\mathbb{R}^{d\cdot n} := \mathbb{R}^d \times \ldots \times \mathbb{R}^d$ (n times) representing the n-particle configuration space, $n = 1, \ldots, N$:

$$\Lambda_L(u) = \mathop{\times}_{1\leq i\leq d} \left[\mathrm{u}^{(i)} - L, \mathrm{u}^{(i)} + L\right], \text{ and } \mathbf{\Lambda}_L(\mathbf{u}) = \mathop{\times}_{1\leq j\leq n} \Lambda_L(u_j), \tag{1.3}$$

where $u = \left(\mathrm{u}^{(1)}, \ldots, \mathrm{u}^{(d)}\right) \in \mathbb{R}^d (= \mathbb{R}^{d\cdot 1})$ and $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{R}^{d\cdot n}$. For our purpose, it suffices to consider only cubes centered at lattice points $u \in \mathbb{Z}^d$ and $\mathbf{u} \in \mathbb{Z}^{d\cdot n} := \mathbb{Z}^d \times \ldots \times \mathbb{Z}^d$. With few exceptions, boldface symbols indicate n-particle objects, related to $\mathbb{R}^{d\cdot n}$ or $\mathbb{Z}^{d\cdot n}d$; a notable exceptions is the symbol $\mathbf{1_A}$ standing for an indicator function of a set $\mathbf{A} \subset \mathbb{R}^{d\cdot n}$ and – with a slight abuse of notation – for the operator of multiplication by this function. It is technically convenient to use the max-norm for vectors $x = \left(\mathrm{x}^{(1)}, \ldots, \mathrm{x}^{(d)}\right) \in \mathbb{R}^d$ and $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^{d\cdot n}$:

$$|x| = \max_{1\leq i\leq d} |x_i|, \quad |\mathbf{x}| = \max \left[\, |x_1|, \ldots, |x_n| \,\right]. \tag{1.4}$$

Correspondingly, the distance dist figuring in the sequel refers to to this norm. In terms of the max-norm a cube $\Lambda_L(u)$ coincides with the ball in $\mathbb{R}^d$ of radius L about point u and $\mathbf{\Lambda}_L(\mathbf{u})$ gives the ball in $\mathbb{R}^{d\cdot n}$ of radius L about point $\mathbf{u}$.

We will also need lattice cubes:

$$B_L(u) = \Lambda_L(u) \cap \mathbb{Z}^d, \quad \text{and} \quad \mathbf{B}_L(\mathbf{u}) = \mathbf{\Lambda}_L(\mathbf{u}) \cap \mathbb{Z}^{d\cdot n}. \tag{1.5}$$

and unit cells (or simply cells):

$$C(u) = \Lambda_1(u) \subset \mathbb{R}^d, \quad \text{and} \quad \mathbf{C}(\mathbf{u}) = \mathbf{\Lambda}_1(\mathbf{u}) \subset \mathbb{R}^{d\cdot n}. \tag{1.6}$$

In what follows, all these sets are also often called boxes, single-particle for $\Lambda_L(u)$, $B_L(u)$ and $C(u)$ and n-particle for $\mathbf{\Lambda}_L(\mathbf{u})$, $\mathbf{B}_L(\mathbf{u})$ and $\mathbf{C}(\mathbf{u})$. A union of cells is referrred to as a cellular set.

1C. Interaction and external field potentials. The interaction potential U in Eqn (1.1) is assumed to be of the form

$$U(\mathbf{x}) = \sum_{k=1,\ldots,N} \sum_{1\leq i_1<\ldots<i_k\leq N} \Phi^{(k)}(x_{i_1}, \ldots, x_{i_k}), \quad \mathbf{x} = (x_1, \ldots, x_N) \in \mathbb{R}^{d\cdot N}, \tag{1.7}$$

where function $\Phi^{(k)} : \mathbb{R}^{d\cdot k} \to \mathbb{R}$, $1 \leq k \leq N$, represents a k-body interaction potential and satisfies the following property:

(I) *Upper-boundedness, non-negativity, symmetry and finite range*:

$$\begin{aligned}
&\forall\ k = 1, \ldots, N: && 0 \leq \Phi^{(k)}(\mathbf{y}) \leq \mathrm{u}_0,\ \mathbf{y} = (y_1, \ldots, y_k) \in \mathbb{R}^{d\cdot k}, \\
&\forall\ k = 2, \ldots, N: && \Phi^{(k)}(y_1, \ldots, y_k) = \Phi^{(k)}(y_{\sigma(1)}, \ldots, y_{\sigma(k)}), \\
& && \forall\ \text{permutation } \sigma \text{ on } \{1, \ldots, k\}, \\
&\forall\ k = 2, \ldots, N: && \Phi^{(k)}(\mathbf{y}) = 0 \text{ when } \max_{1\leq i\leq N} \left(\min_{j\neq i} \left[\, |y_i - y_j| \,\right]\right) \geq \mathrm{r}_0,
\end{aligned} \tag{1.8}$$

where constants $\mathrm{u}_0, \mathrm{r}_0 \in (0, +\infty)$.

Remark 1.1. Non-negativity of potentials $\Phi^{(k)}$ is used to simplify the statement of the main result (see Theorem 1.1 below) and shorten the proof of technical assertions. We can also relax the boundedness condition, by allowing hard-core potentials where, $\forall\ k \geq 2$, $\Phi^{(k)}(y_1, \dots, y_k) = +\infty$ whenever $\min\limits_{1\leq i<j\leq k} \left|y_i - y_j\right| < \mathrm{r}_1$ where $\mathrm{r}_1 \in (0, \mathrm{r}_0)$. On the other hand, the finite-range condition is essential for the method used, and extending our Theorem 1.1 to the case of infinite-range potentials seems a challenging problem.

Further, the random external potential $V(x;\omega)$, $x \in \mathbb{R}^d$, $\omega \in \Omega$, is assumed to be of alloy-type, over a cubic lattice:

$$V(x;\omega) = \sum_{s\in\mathbb{Z}^d} \mathrm{V}_s(\omega)\varphi_s(x-s). \tag{1.9}$$

Here $\mathcal{V} = (\mathrm{V}_s,\ s \in \mathbb{Z}^d)$, is a family of real random variables V_s on some probability space $(\Omega, \mathfrak{B}, \mathbb{P})$ and $\{\varphi_s,\ s \in \mathbb{Z}^d\}$ is a (nonrandom) collection of 'bump' functions $y \in \mathbb{R}^d \mapsto \varphi_s(y)$. In probabilistic terms, $\mathcal{V}$ is a real-valued random field (RF) on $\mathbb{Z}^d$. Physically speaking, the random variable V_s represents the amplitude of 'impurity' at site s of lattice $\mathbb{Z}^d$ while the function φ_s describes the 'propagation' of the impact of this impurity across $\mathbb{R}^d$.

In this paper we do not use independence of random variables V_s for different sites $s \in \mathbb{Z}^d$. However, we impose conditions **(V1)**–**(V3)** below.

(V1) *Upper-boundedness and non-negativity*:

$$\operatorname*{supess}_{\mathcal{V}} \sup_{s\in\mathbb{Z}^d} \mathrm{V}_s < \infty, \quad \operatorname*{infess}_{\mathcal{V}} \inf_{s\in\mathbb{Z}^d} \mathrm{V}_s \geq 0 \tag{1.10}$$

Remark 1.2. Again, non-negativity plays a technical role and is not crucial for the main result. The boundedness condition for random variables V_s can be replaced by finiteness of expectations $\mathbb{E}\left(|\mathrm{V}_s|^{\mathrm{r}}\right)$ for some $\mathrm{r} > 0$.

Given a site $s \in \mathbb{Z}^d$, consider the conditional distribution function

$$F\left(\mathrm{y}\middle|\mathfrak{B}^{\mathrm{c}}_s\right) := \mathbb{P}\big(\mathrm{V}_s < \mathrm{y}\big|\mathfrak{B}^{\mathrm{c}}_s\big), \quad y \in \mathbb{R}, \tag{1.11}$$

relative to the sigma-algebra $\mathfrak{B}^{\mathrm{c}}_s$ generated by the family $\mathcal{V}_s = \left\{\mathrm{V}_t,\ t \in \mathbb{Z}^d \setminus \{s\}\right\}$. Owing to (1.9), function $F\left(\mathrm{y}\middle|\mathfrak{B}^{\mathrm{c}}_s\right)$ vanishes for $y < 0$ for $\mathbb{P}$-almost all conditions. We assume

(V2) *Uniform Hölder-continuity of* $F\left(\mathrm{y}\middle|\mathfrak{B}^{\mathrm{c}}_s\right)$: for some $a, b > 0$ and all $\epsilon \in (0,1)$,

$$\nu(\epsilon) := \sup_{s\in\mathbb{Z}^d} \sup_{\mathrm{y}\in\mathbb{R}} \operatorname*{sup\,ess}_{\mathcal{V}_{\{s\}^{\mathrm{c}}}} \left[F\left(\mathrm{y}+\epsilon\middle|\mathfrak{B}^{\mathrm{c}}_s\right) - F\left(\mathrm{y}\middle|\mathfrak{B}^{\mathrm{c}}_s\right)\right] \leq a\epsilon^b. \qquad (1.12).$$

Remark 1.3. Conditions **(V1)**–**(V2)** cover a wide enough class of examples, such as $\mathrm{V}_s = \big(\cos\ \mathrm{W}_s\big)^2$ where $\mathcal{W} = \big(\mathrm{W}_s,\ s \in \mathbb{Z}^d\big)$ is a zero-mean non-degenerate Gaussian random field over $\mathbb{Z}^d$. Another type of examples satisfying these conditions can be found among Gibbs random fields on $\mathbb{Z}^d$. On the other hand, the main result of this paper remains valid under a much weaker assumption of log-Hölder continuity: $\nu(\epsilon) \le a\big|\ln\ \epsilon\big|^{-b}$, for $b > 0$ large enough. We would also like to note that in [CS09A], in a context of a tight-binding Anderson model, a stronger assumption was made, that (i) random variables V_s are independent and identically distributed, and (ii) each V_s has a bounded probability density function p_{V} of compact support. This assumption was used when we applied results from Ref. [A94] in the proof of Lemma 5.1 from [CS09A]. However, in a subsequent work [CS09B] an alternative argument has been produced, requiring broader assumptions that are close to those used in the present paper.

Our last group of conditions, **(F1)** – **(F2)**, is imposed on the collection of bump functions $\{\varphi_s,\ s \in \mathbb{Z}^d\}$. First, we need

(F1) *Boundedness, non-negativity and compact support of* φ_s: the bump functions φ_s are non-negative functions, with bounded support, such that

$$\sup_{x\in\mathbb{R}^d}\left[\sum_{s\in\mathbb{Z}^d}\ \varphi_s(x-s)\right] < +\infty,\ \ \forall\ x \in \mathbb{R}^d. \tag{1.13}$$

and $\exists\ \mathrm{r}^{(0)} \in (0,\infty)$ with

$$\varphi_s(y) = 0\ \ \text{whenever}\ \ |y| > \mathrm{r}^{(0)},\ y \in \mathbb{R}^d. \tag{1.14}$$

We will also use

(F2) *Covering condition for* φ_s:

$$\sum_{s\in\Lambda_L(u)\cap\mathbb{Z}^d} \varphi_s(x-s) \ge\ 1,\ \ \forall\ L \ge 1,\ u \in \mathbb{R}^d,\ x \in \Lambda_L(u). \tag{1.15}$$

Remark 1.4. As above, Assumptions **(F1)-(F2)** play a technical role and can be relaxed.

1D. Main result. The main result of this paper is the following Theorem 1.1. All properties listed in this theorem hold with $\mathbb{P}$-probability one.

Theorem 1.1. *Consider the operator* $\mathbf{H}$ *from* (1.1). *Under conditions* **(U)**, **(V1)**–**(V2)** *and* **(F1)**–**(F2)**, *it admits a unique self-adjoint extension from the set of* C^2*-functions with compact support in* $\mathbb{R}^{d\cdot N}$. *This self-adjoint extension, again denoted by* $\mathbf{H}$, *is a random positive-definite operator with the following property. Let* $E_0^* \ge 0$ *be the lower edge of the spectrum of the operator* $-\dfrac{1}{2}\mathbf{\Delta} + \mathbf{U}(\mathbf{x})$ *(the Hamiltonian in absense of the random external potential). There exists a non-random value* $E_1^* > E_0^*$ *such that the spectrum of* $\mathbf{H}$ *in the interval* $[E_0^*, E_1^*]$ *is pure point. Furthermore, there exists a non-random constant* $m^* > 0$ *such that for each eigenfunction* $\mathbf{\Psi}_j(\mathbf{x};\omega)$ *with eigenvalue* $E_j \in [E_0^*, E_1^*)$ *and* $\forall\ \mathbf{v} \in \mathbb{Z}^{d\cdot N}$, *the norm* $\|\mathbf{1}_{\mathbf{C}(\mathbf{v})}\mathbf{\Psi}_j(\ \cdot\ ;\omega)\|$ *of the projected vector* $\mathbf{1}_{\mathbf{C}(\mathbf{v})}\mathbf{\Psi}_j(\ \cdot\ ;\omega)$ *obeys*

$$\|\mathbf{1}_{\mathbf{C}(\mathbf{v})}\mathbf{\Psi}_j(\cdot;\omega)\| \le C_j e^{-m^*|\mathbf{v}|}. \tag{1.16}$$

where $C_j = C_j(\omega) \in (0,+\infty)$ *is a random constant varying with* j.

Here and below, $\| \ \|$ stands for the norm (of a vector or an operator) in $\mathcal{L}_2(\mathbb{R}^{d\cdot N})$ or in $\mathcal{L}_2(\mathbb{R}^{d\cdot n})$, $1 \le n \le N$, as specifed by the local context.

Remarks. 1.5. Constant m^* is often referred to as an 'effective mass' (or briefly a 'mass') in Hamiltonian $\mathbf{H}$. From the physical point of view, the statement of Theorem 1.1 is in agreement with the so-called Lifshits tail theory.

1.6. The spectrum of operator $\mathbf{H}$ may have an empty intersection with $[E_0^*, E_1^*]$; in this case the assertion of Theorem 1.1 is satisfied automatically. To exclude such a case, one could assume that point 0 belongs to the support of the law of each variable V_s, more precisely, that the conditional distribution function $F\left(\mathrm{y}\middle|\mathfrak{B}_s^{\mathrm{c}}\right)$ in (1.9) is strictly monotone in $\mathrm{y} \in [0, \delta]$ for some $\delta > 0$.

1.7. It is worth observing the following fact. Suppose the one-body potential $\Phi^{(1)}$ in (1.7) (which can be considered as a 'non-random' part of the external field) is constant: $\Phi^{(1)}(x) \equiv a$, $x \in \mathbb{R}^d$. Then, under the finite-range condition (1.8), the essential spectrum of operator $-\frac{1}{2}\mathbf{\Delta} + \mathbf{U}(\mathbf{x})$ begins at Na (i.e., $E_0^* = Na$). This is because there are configurations $\mathbf{x} = (x_1, \dots, x_N) \in \mathbb{R}^{d\cdot N}$ where $U(\mathbf{x})$ is reduced to the sum $\sum_{1\le j\le N} \Phi^{(1)}(x_j) = Na$. Also recall that, by virtue of a result from [KZ03], the integrated density of states for the (non-random) operator $-\frac{1}{2}\mathbf{\Delta} + \mathbf{U}(\mathbf{x})$ is the same as for $\mathbf{H}$.

In what follows we focus on the property that the spectrum of $\mathbf{H}$ in $[E_0^*, E_1^*]$ is pure point and on the inequality (1.16); the preceding statements of Theorem 1.1 are straightforward. Throughout the paper we will assume the conditions of Theorem 1.1 although some constructions used below remain valid under broader assumptions.

1E. Reduction to the MSA bound (1.24). The proof of Theorem 1.1 is based on the analysis of the operators $\mathbf{H}^{\mathbf{\Lambda}}$, the finite-volume versions of $\mathbf{H}$. More precisely, let $\mathbf{\Lambda} = \mathbf{\Lambda}_L(\mathbf{u})$ and consider the operator $\mathbf{H}^{\mathbf{\Lambda}}$ in $\mathcal{L}_2(\mathbf{\Lambda})$ defined as in (1.1):

$$\mathbf{H}^{\mathbf{\Lambda}} = -\frac{1}{2}\mathbf{\Delta}^{\mathbf{\Lambda}} + \mathbf{U}(\mathbf{x}) + \mathbf{V}(\omega; \mathbf{x}) \tag{1.17}$$

where $\mathbf{\Delta}^{\mathbf{\Lambda}}$ stands for the kinetic energy operator in $\mathcal{L}_2(\mathbf{\Lambda})$ with Dirichlet's boundary conditions on $\partial\mathbf{\Lambda}$. Under assumptions (D) and (E1)–(E4), there exists a unique self-adjoint extension of $\mathbf{H}^{\mathbf{\Lambda}}$ from the set of C^2-functions vanishing in a neighbourhood of the boundary $\partial\mathbf{\Lambda}$; we again denote it by $\mathbf{H}^{\mathbf{\Lambda}}$. Then $\mathbf{H}^{\mathbf{\Lambda}}$ is a random positive-definite operator with pure point spectrum $\mathbf{\Sigma}\left(\mathbf{H}^{\mathbf{\Lambda}}\right) \subset [0, +\infty)$. Furthermore, the resolvent $\mathbf{G}^{\mathbf{\Lambda}}(E) = (\mathbf{H}^{\mathbf{\Lambda}} - E\mathbf{I})^{-1}$, for $E \in \mathbb{C} \setminus \mathbf{\Sigma}\left(\mathbf{H}^{\mathbf{\Lambda}}\right)$, is a compact integral operator in $\mathcal{L}_2(\mathbf{\Lambda})$. Probabilistically, the random eigenvalues and eigenvectors of $\mathbf{H}^{\mathbf{\Lambda}}$ are measurable relative to the sigma-algebra $\mathfrak{B}_{\overline{\mathbf{B}}_{\mathrm{r}^{(0)}}}$ generated by the family $\mathcal{V}_{\overline{\mathbf{B}}_{\mathrm{r}^{(0)}}}$. The latter is formed by random variables V_s with $\mathrm{dist}\,[s, \mathbf{\Lambda}] := \min\,\left[|s - y| : \ y \in \mathbf{\Lambda}\right] \le \mathrm{r}^{(0)}$ where $\mathrm{r}^{(0)}$ is the constant from (1.14).

For $L > 2$ define the outer layer $\mathbf{\Lambda}_L^{\mathrm{out}}(\mathbf{u})$ in a box $\mathbf{\Lambda}_L(\mathbf{u})$ by

$$\mathbf{\Lambda}_L^{\mathrm{out}}(\mathbf{u}) = \mathbf{\Lambda}_L(u) \setminus \mathbf{\Lambda}_{L-2}(u), \ \ \mathbf{u} \in \mathbb{Z}^{d\cdot N}. \tag{1.18}$$

Definition 1.1. *Given $E \in \mathbb{R}$, $m > 0$ and $\mathbf{u} \in \mathbb{Z}^{d\cdot N}$, the N-particle box $\mathbf{\Lambda}_L(\mathbf{u})$ is called (E,m)-non-singular (briefly, (E,m)-NS), if for any $\mathbf{v} \in \mathbf{\Lambda}_L^{\rm out}(\mathbf{u}) \cap \mathbb{Z}^{d\cdot N}$, the $\mathcal{L}_2(\mathbf{\Lambda}_L(\mathbf{u}))$-norm of the vector*

$$\mathbf{1}_{\mathbf{C}(\mathbf{u})}\, \mathbf{G}^{\mathbf{\Lambda}_L(\mathbf{v})}(E)\, \mathbf{1}_{\mathbf{C}(\mathbf{w})}(\mathbf{x}) := \mathbf{1}_{\mathbf{C}(\mathbf{u})}(\mathbf{x}) \left[\mathbf{G}^{\mathbf{\Lambda}_L(\mathbf{v})}(E)\, \mathbf{1}_{\mathbf{C}(\mathbf{w})} \right](\mathbf{x}) \tag{1.19}$$

admits the bound

$$\left\| \mathbf{1}_{\mathbf{C}(\mathbf{v})}\, \mathbf{G}^{\mathbf{\Lambda}_L(\mathbf{v})}(E)\, \mathbf{1}_{\mathbf{C}(\mathbf{w})} \right\| \le e^{-\gamma(m,L)}, \tag{1.20}$$

where

$$\gamma(m,L) := mL \left(1 + L^{-1/4}\right). \tag{1.21}$$

Otherwise, $\mathbf{\Lambda}_L(\mathbf{u})$ is called (E,m)-singular ((E,m)-S). When the reference to values E and m can be omitted, we speak of simply of S-boxes.

Definition 1.2. *Let $\mathrm{R} > 0$ and $\mathbf{u} = (u_1,\dots,u_N), \mathbf{v} = (v_1,\dots,v_N) \in \mathbb{Z}^{d\cdot N}$. A pair of N-particle boxes $\mathbf{\Lambda}_L(\mathbf{u})$, $\mathbf{\Lambda}_L(\mathbf{v})$ is called R-distant if, $\forall$ permutation σ on $\{1,\dots,N\}$,*

$$|\mathbf{u} - \sigma(\mathbf{v})| > 8\mathrm{R} \quad \text{where} \quad \sigma(\mathbf{v}) = (v_{\sigma(1)},\dots,v_{\sigma(N)}). \tag{1.22}$$

The N-particle MSA scheme deduces Theorem 1.1 from the following Theorems 1.2 and 1.3.

Theorem 1.2. *Fix $\alpha > 1$ and $p > \alpha d$. Given $L_0 > 1$, set:*

$$L_k = L_0^{\alpha^k}, \quad k = 1,2,\dots. \tag{1.23}$$

Suppose that for some $E_0 < E_1$, $m > 0$ and $L_0 > 0$, $\forall\, k \ge 0$ the following bound holds true: for any pair of L_k-distant boxes $\mathbf{\Lambda}_{L_k}(\mathbf{u}')$ and $\mathbf{\Lambda}_{L_k}(\mathbf{u}'')$,

$$\mathbb{P}\left\{ \exists\, E \in [E_0,E_1]:\ \mathbf{\Lambda}_{L_k}(\mathbf{u}') \text{ and } \mathbf{\Lambda}_{L_k}(\mathbf{u}'') \text{ are } (E,m)\text{-S} \right\} \le L_k^{-2p}. \tag{1.24}$$

Then with $\mathbb{P}$-probability one, the spectrum of operator $\mathbf{H}$ (see (1.1) in interval $[E_0,E_1]$ is pure point. Furthermore, $\exists\ m^* > 0$ such that every eigenfunction $\mathbf{\Psi}_j(\mathbf{x};\omega)$ of $\mathbf{H}$ with the eigenvalue $E_j(\omega) \in [E_0,E_1]$ satisfies Eqn (1.16).

Theorem 1.3. *Let E_0^* be as in Theorem 1.1. Given $\alpha > 1$ and $p > \alpha d$, there exist $L_0 > 1$, $m > 0$ and $E_1^* > E_0^*$ such that, for L_k defined in Eqn (1.22), the bound (1.24) holds true, with $E_0 = E_0^*$, $E_1 = E_1^*$.*

Remark 1.7. Definition 1.1 has been inspired by [DK89], P. 287; see also Definition 1 from [CS09A] and Definition 1.1 from [CS09B]. However, the reader familiar with the MSA would note a difference resulting in using a bound by $e^{-\gamma(m,L)}$ instead of more traditional e^{-mL}. It allows us to avoid a (rather tedious) procedure of re-scaling the mass m_k when we pass from length L_k to L_{k+1} as defined in (1.23). Cf. [DK89], Lemma 4.1, or [CS09A], Eqn (1.12) and [CS09B], Eqns (1.12). Indeed, it is straightforward that, when positive numbers m_k and m_{k+1} are tied by $m_{k+1} \ge m_k(1 - L_k^{-1/2})$, then

$$\begin{aligned}&\gamma(m_k,L_k)(1 - L_k^{-1/2}) = m_k(1 + L_k^{-1/4})(1 - L_k^{-1/2})\\ &= m_k(1 + L_k^{-1/4} - L_k^{-1/2} - L_k^{-1/8}) > m_k(1 + L_{k+1}^{-1/4}) = \gamma(m_k,L_{k+1}),\end{aligned}$$

provided that L_k is large enough, so that $L_k^{1/2} - 2 > L_k^{1/8}$. Therefore, having a decay exponent $\gamma(m, L_k)$ at scale L_k, a 'standard' rescaling gives a decay exponent *larger* than $\gamma(m, L_{K+1})$ at the next scale L_{k+1}. It means that we will be able to use the decay exponent $\gamma(m, L_{k+1})$ without re-scaling the value of the parameter m: function $\gamma(m, L)$ automatically takes care of it.

1F. The plan for the rest of the paper. In Section 2, we discuss resolvent inequalities – the main technical tool in the proof of Theorems 1.2 and 1.3. This section is, in a sense, a core of the whole paper. From there on, we are able to employ the multi-particle MSA scheme from [CS09A], [CS09B].

Consequently, in Section 3 we give the proof of Theorem 1.2, closely following the argument from Section 2 of [CS09A]. (In fact, this argument goes back to [DK89].)

Next, in Section 4 we prove Theorem 1.3, employing – with necessary modifications – the arguments from Sections 3–5 of [CS09A]. In particular, as in [CS09A], in the course of the proof we check the assertion of of Theorem 1.3 separately for three types of pairs of distant and singular N-particle boxes: (i) for pairs of non-interactive boxes, (ii) for pairs of interactive boxes and (iii) for pairs where one of the boxes is interactive and the second non-interactive. (The terminology follows [CS09A], [CS09B] and is formally explained in due course.) This is carried out in sub-Sections 4(i), 4(ii) and 4(iii), respectively. Note that in sub-Section 4(i) we use a new argument that is simpler than that from Section 3 in [CS09A]: this became possible due to a specific form of the tunneling property (already used in [CS09B]).

Finally, in a (short) Section 5 we give a (straightforward) proof of a technical lemma used in sub-Section 4(i).

2. Resolvent inequalities

Along with Hamiltonian $\mathbf{H}^{\mathbf{\Lambda}}$ in an N-particle box $\mathbf{\Lambda} \subset \mathbb{R}^{d\cdot N}$ we will consider its n-particle counterpart where $n \in \{1, \dots, N\}$, following the same definition (1.17) with obvious modifications. E.g., in a single-particle Hamiltonian the term $\mathbf{U}(\mathbf{x})$ is reduced to $\Phi^{(1)}(x)$ and the external field $\mathbf{V}(\omega; \mathbf{x})$ to $V(\omega; x)$. It will be convenient to use the common notation $\mathbf{H}^{\mathbf{\Lambda}}$ indicating, when necessary, that $\mathbf{\Lambda} \subset \mathbb{R}^{d\cdot n}$ is an n-particle box. In particular, Definitions 1.1 and 1.2 are carried through for n-particle Hamiltonians (and Theorems 1.1–1.3 will be established for n-particle systems) $\forall\ n = 1, \dots, N$. Furthermore, all constructions and definitions inroduced below can be repeated, *mutatis mutandis*, for n replacing N.

A number of constructions below will revolve around the following definition:

Definition 2.1. *Set $\beta = 1/2$ and fix $\alpha > 1$. Given $E \in \mathbb{R}$ and $\mathbf{u} \in \mathbb{Z}^{d\cdot N}$, the N-particle box $\mathbf{\Lambda}_L(\mathbf{u})$ is called E-non-resonant (E-NR, in short) if for any $\ell \in \left[L^{1/\alpha}, L\right)$ and any N-particle box $\mathbf{\Lambda}_\ell(\mathbf{v}) \subseteq \mathbf{\Lambda}_L(\mathbf{u})$, the following bound holds true:*

$$\mathrm{dist}[E, \mathbf{\Sigma}\left(\mathbf{H}^{\mathbf{\Lambda}_\ell(\mathbf{v})}\right)] \geq \mathrm{e}^{-\ell^\beta}. \tag{2.1}$$

Otherwise, $\mathbf{\Lambda}_L(\mathbf{u})$ is called E-resonant (E-R).

As is well-understood by now, the MSA is based on (i) a certain number of probabilistic-type bounds, proved either (i1) inductively in parameter k from Eqn (1.23) (viz., decay estimates related to resolvent $\mathbf{G}^{\mathbf{\Lambda}_{L_k}(\mathbf{u})}(E)$) or (i2) for all scales L_k at once (e.g., Wegner-type bounds; see below), combined with (ii) "deterministic", functional-analytic-type inequalities (again related to resolvents $\mathbf{G}^{\mathbf{\Lambda}_{L_k}(\mathbf{u})}(E)$). In this section, we discuss latter-type inequalities; our aim is to show that these can be essentially reduced to bounds for some auxiliary functions defined on lattice $\mathbb{Z}^{d\cdot N}$.

To this end, consider two embedded N-particle boxes, $\mathbf{\Lambda} \subset \widetilde{\mathbf{\Lambda}}$ where $\mathbf{\Lambda} = \mathbf{\Lambda}_L(\mathbf{u})$ and $\widetilde{\mathbf{\Lambda}} = \mathbf{\Lambda}_{\widetilde{L}}(\widetilde{\mathbf{u}})$, with $4 \leq L < \widetilde{L}$, and set

$$\mathbf{\Lambda}^{\rm int} := \mathbf{\Lambda}_{L/3}(\mathbf{u}). \tag{2.2}$$

Let $\mathcal{A} \subset \mathbf{\Lambda}^{\rm int}$ and $\mathcal{B} \subset \widetilde{\mathbf{\Lambda}} \setminus \mathbf{\Lambda}$. For our purposes, it suffices to assume that sets $\mathcal{A}$ and $\mathcal{B}$ are cellular. The standard resolvent identity for (dN)-dimensional Schrödinger operators, combined with commutator estimates, implies the following geometric resolvent inequality (GRI). Given a $a \in \mathbb{R}$ and $\eta \in (0, +\infty)$, form the interval $I = [a - \eta/2, a + \eta/2] \subset \mathbb{R}$. Then for $\forall\, E \in I \setminus \Big(\mathbf{\Sigma}\,(\mathbf{H}^{\mathbf{\Lambda}}) \cup \mathbf{\Sigma}\,(\mathbf{H}^{\widetilde{\mathbf{\Lambda}}})\Big)$:

$$\textbf{(GRI):}\ \ \|\mathbf{1}_{\mathcal{B}}\, \mathbf{G}^{\widetilde{\mathbf{\Lambda}}}(E)\, \mathbf{1}_{\mathcal{A}}\| \leq \mathrm{c}\|\mathbf{1}_{\mathcal{B}}\, \mathbf{G}^{\widetilde{\mathbf{\Lambda}}}(E)\, \mathbf{1}_{\mathbf{\Lambda}^{\rm out}}\|\, \|\mathbf{1}_{\mathbf{\Lambda}^{\rm out}}\, \mathbf{G}^{\mathbf{\Lambda}}(E)\, \mathbf{1}_{\mathcal{A}}\|. \tag{2.3}$$

Here $\mathrm{c} > 0$ is a 'geometric' constant: owing to condition $4 \leq L < \widetilde{L}$, it only depends on the product dN and values a and η. Cf. [St01], Lemma 2.5.4.

Bound (2.3) enables us to use 'discretization' of some important functions related to resolvents $G^{\mathbf{\Lambda}}(E)$ and $G^{\mathbf{\Lambda}'}(E)$ and defined originally in the continuous space $\mathbb{R}^{d\cdot N}$ and to reduce most of necessary estimates to functions defined on the lattice $\mathbb{Z}^{d\cdot N}$. This leads to a unified approach to Anderson localization in both discrete and continuous settings.

Remark 2.1. The methods outlined above admit a natural extension to other d-dimensional lattices $\mathcal{Z} \subset \mathbb{R}^d$, i.e. additive subgroups $\mathcal{Z} \subset \mathbb{R}^d$ generated by d linearly independent vectors $\mathbf{e_1}, \ldots, \mathbf{e_d} \in \mathbb{R}^d$.

2A. Discretized integrated Green's functions. Given $\widetilde{\mathbf{u}} \in \mathbb{Z}^{d\cdot N}$ and $\widetilde{L} > 7$, consider the box $\widetilde{\mathbf{\Lambda}} = \mathbf{\Lambda}_{\widetilde{L}}(\widetilde{\mathbf{u}})$ and its lattice counterpart $\widetilde{\mathbf{B}} = \widetilde{\mathbf{\Lambda}} \cap \mathbb{Z}^{d\cdot N}$. Further, pick a point $\mathbf{u} \in \widetilde{\mathbf{B}}$ and a number $L \in (3, \widetilde{L} - 3)$ such that the box $\mathbf{\Lambda} = \mathbf{\Lambda}_L(\mathbf{u})$ lies in $\mathbf{\Lambda}_{\widetilde{L}-3}(\widetilde{\mathbf{u}})$. Like above (cf. Eqn (1.18)), set:

$$\widetilde{\mathbf{\Lambda}}^{\rm out} = \widetilde{\mathbf{\Lambda}} \setminus \mathbf{\Lambda}_{\widetilde{L}-2}(\widetilde{\mathbf{u}}), \quad \mathbf{\Lambda}^{\rm out} = \mathbf{\Lambda} \setminus \mathbf{\Lambda}_{L-2}(\mathbf{u}), \tag{2.4.1}$$

and

$$\widetilde{\mathbf{B}}^{\rm out} = \widetilde{\mathbf{\Lambda}}^{\rm out} \cap \mathbb{Z}^{d\cdot N}, \quad \mathbf{B}^{\rm out} = \mathbf{\Lambda}^{\rm out} \cap \mathbb{Z}^{d\cdot N}. \tag{2.4.2}$$

It is clear that

$$\widetilde{\mathbf{\Lambda}}^{\rm out} \subset \bigcup_{\mathbf{v} \in \widetilde{\mathbf{B}}^{\rm out}} \mathbf{C}(\mathbf{v}), \quad \mathbf{\Lambda}^{\rm out} \subset \bigcup_{\mathbf{v} \in \mathbf{B}^{\rm out}} \mathbf{C}(\mathbf{v}),$$

so that for the indicator functions

$$\mathbf{1}_{\widetilde{\mathbf{\Lambda}}^{\rm out}}(\mathbf{x}) \leq \sum_{\mathbf{v} \in \widetilde{\mathbf{B}}^{\rm out}} \mathbf{1}_{\mathbf{C}(\mathbf{v})}(\mathbf{x}), \quad \mathbf{1}_{\mathbf{\Lambda}^{\rm out}}(\mathbf{x}) \leq \sum_{\mathbf{v} \in \mathbf{B}^{\rm out}} \mathbf{1}_{\mathbf{C}(\mathbf{v})}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^{d\cdot N}. \tag{2.5}$$

Therefore, bound (2.3) implies that for any $\mathbf{w} \in \widetilde{\mathbf{B}}^{\rm out}$, the following inequality holds true: $\forall\, E \in I \setminus \Big(\mathbf{\Sigma}\left(\mathbf{H}^{\mathbf{\Lambda}}\right) \cup \mathbf{\Sigma}\left(\mathbf{H}^{\mathbf{\Lambda}'}\right) \Big)$:

$$
\begin{aligned}
\textbf{LGRI: } & \|\mathbf{1}_{\mathbf{C}(\mathbf{u})}\, \mathbf{G}^{\widetilde{\mathbf{\Lambda}}}(E)\, \mathbf{1}_{\mathbf{C}(\mathbf{w})}\| \\
& \le \mathrm{c} \sum_{\mathbf{v} \in \mathbf{B}^{\rm out}} \|\mathbf{1}_{\mathbf{C}(\mathbf{u})}\, \mathbf{G}^{\widetilde{\mathbf{\Lambda}}}(E)\, \mathbf{1}_{\mathbf{C}(\mathbf{v})}\|\, \|\mathbf{1}_{\mathbf{C}(\mathbf{v})}\, \mathbf{G}^{\mathbf{\Lambda}}(E)\, \mathbf{1}_{\mathbf{C}(\mathbf{w})}\|.
\end{aligned}
\tag{2.6.1}
$$

We call Eqn (2.6) the *lattice geometric resolvent inequality* (LGRI, for short), in order to distinguish it from the GRI in (2.3). It is useful to remember that $\mathbf{u}$ is the centre of box $\mathbf{\Lambda}$ and should be associated with the 'inner' subset $\mathcal{A}$ in Eqn (2.3).

It is instructive to re-write eqn (2.6.1) as

$$
\mathcal{R}_{\widetilde{L},\widetilde{\mathbf{u}}}(\mathbf{u},\mathbf{w};E) \le \mathrm{c} \sum_{\mathbf{v} \in \mathbf{B}^{\rm out}} \mathcal{R}_{L,\mathbf{u}}(\mathbf{u},\mathbf{v};E)\mathcal{R}_{\widetilde{L},\widetilde{\mathbf{u}}}(\mathbf{v},\mathbf{w};E). \tag{2.6.2}
$$

Here $\mathcal{R}_{L,\mathbf{u}}(\,\cdot\,,\,\cdot\,;E)$ and $\mathcal{R}_{\widetilde{L},\widetilde{\mathbf{u}}}(\,\cdot\,,\,\cdot\,;E)$ are given by

$$
\begin{aligned}
\mathcal{R}_{L,\mathbf{u}}(\mathbf{x},\mathbf{x}';E) &:= \|\mathbf{1}_{\mathbf{C}(\mathbf{x})}\, \mathbf{G}^{\mathbf{\Lambda}_L(\mathbf{u})}(E)\, \mathbf{1}_{\mathbf{C}(\mathbf{x}')}\|, \;\; \mathbf{x},\mathbf{x}' \in \mathbf{B}_L(\mathbf{u}),\\
\mathcal{R}_{\widetilde{L},\widetilde{\mathbf{u}}}(\mathbf{y},\mathbf{y}';E) &:= \|\mathbf{1}_{\mathbf{C}(\mathbf{y})}\, \mathbf{G}^{\mathbf{\Lambda}_{\tilde{L}}(\widetilde{\mathbf{u}})}(E)\, \mathbf{1}_{\mathbf{C}(\mathbf{y}')}\|, \;\; \mathbf{y},\mathbf{y}' \in \mathbf{B}_{\widetilde{L}}(\widetilde{\mathbf{u}}).
\end{aligned}
\tag{2.7}
$$

Functions $\mathcal{R}_{L,\mathbf{u}}(\,\cdot\,,\,\cdot\,;E)$ and $\mathcal{R}_{\widetilde{L},\widetilde{\mathbf{u}}}(\,\cdot\,,\,\cdot\,;E)$ can be called discretized integrated Green's functions (for operators $\mathbf{H}^{\mathbf{\Lambda}}$ and $\mathbf{H}^{\widetilde{\mathbf{\Lambda}}}$, respectively).

Now the analogy with the lattice version of the resolvent inequality is straightforward (cf., e.g., Eqn (4.1) in [DK89]); the only difference is a 'geometric' constant c in the RHS. However, with the factors $\mathcal{R}_{\mathbf{\Lambda}}(\mathbf{u},\mathbf{v};E)$ small enough, this constant will not require a substantial modification of lattice MSA techniques.

Remark 2.2. The reader familiar with the MSA can see now that the central task of the MSA for the alloy-type model in $\mathbb{R}^{d\cdot N}$ considered in this paper is essentially reduced to the analysis of the decay properties of the functions $\mathcal{R}_{L,\mathbf{u}}(\mathbf{v},\mathbf{w};E)$ when $E \in \mathbb{R} \setminus \mathbf{\Sigma}(\mathbf{H}^{\mathbf{\Lambda}_L(\mathbf{u})})$, $L > 0$ is large enough and lattice sites $\mathbf{v}$ and $\mathbf{w}$ are distantr apart (viz., $\mathbf{v}$ is 'deeply' inside $\mathbf{B}_L(\mathbf{u})$ whereas $\mathbf{w}$ is near the boundary of $\mathbf{B}_L(\mathbf{u})$; see below). But of course, it does not mean that the spectral problem for the operators $\mathbf{H}$ and $\mathbf{H}^{\mathbf{\Lambda}}$ is formally reduced to that for a tight-binding Hamiltonians in $\ell_2(\mathbb{Z}^{d\cdot N})$ and $\ell_2(\mathbf{B}_L(\mathbf{u})$.

Remark 2.3. It is worth mentioning that our reduction of the MSA in Euclidean space to an auxiliary lattice problem is not contingent upon a particular structure of the random external potential. The fact that the centers of the scatterers of the alloy-type potential considered in this paper form the same cubic lattice $\mathbb{Z}^d$ as the centers of unit cells $\mathbf{C}(\mathbf{v})$ is a mere coincidence. Moreover, the above mentioned discretization can be used, with no modification, in the case where the random potential $V(x;\omega)$ is a random field with continuous argument (e.g., a regular Gaussian field with continuous argument, as in our recent manuscript [**?**]).

For the rest of this paper, points $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, etc, representing centres of boxes or cells will be assumed to be in $\mathbb{Z}^{d\cdot N}$ without stressing it every time again. Similarly, parameter L is assumed to be a positive integer. While working with a lattice box $\mathbf{B} = \mathbf{B}_L(\mathbf{u}) \subset \mathbb{Z}^{d\cdot N}$ (and, more generally, cellular subsets in $\mathbb{Z}^{d\cdot N}$, we will employ the traditional notation for the inner boundary $\partial^-\mathbf{B}$, exterior boundary $\partial^+\mathbf{B}$, and the 'full' boundary $\partial\mathbf{B}$, defined as follows:

$$\begin{aligned}\partial^-\mathbf{B} &= \left\{ \mathbf{x} :\in \mathbf{\Lambda} : \ \operatorname{dist}[\mathbf{x}, \mathbb{Z}^{d\cdot N} \setminus \mathbf{B}] = 1 \right\},\\ \partial^+\mathbf{B} &= \left\{ \mathbf{x} :\in \mathbb{Z}^{d\cdot N} \setminus \mathbf{\Lambda} : \ \operatorname{dist}[\mathbf{x}, \mathbf{B}] = 1 \right\},\\ \partial\mathbf{B} &= \{ (\mathbf{x}, \mathbf{x}') : |\mathbf{x} - \mathbf{x}'| = 1, \mathbf{x} \in \partial^-\mathbf{B}, \mathbf{x}' \in \partial^+\mathbf{B} \}.\end{aligned} \tag{2.8}$$

2B. The LGRI for NS-boxes. Here the setting is simple: fix an N-particle lattice box $\mathbf{\Lambda}_L(\mathbf{u})$ with $\mathbf{u} \in \mathbb{Z}^{d\cdot N}$, point $E \in \mathbb{C}$ and numbers $m > 0$ and $\ell \in (0, L)$. Assume that $\mathbf{\Lambda}_L(\mathbf{u})$ does not contain any (E, m)-S box $\mathbf{B}_\ell(\mathbf{v})$ with $\mathbf{v} \in \mathbb{Z}^{d\cdot N} \cap \mathbf{\Lambda}_L(\mathbf{u})$. Then the LGRI (2.6) implies that, $\forall$ site $\mathbf{y} \in \mathbb{Z}^{d\cdot N} \cap \partial^-\mathbf{B}_L(\mathbf{u})$ and $\forall$ box $\mathbf{\Lambda}_\ell(\mathbf{v}) \subset \mathbf{\Lambda}_L(\mathbf{u})$, for the norm $\mathcal{R}_{L,\mathbf{u}}(\mathbf{v}, \mathbf{w}; E)$ defined as in Eqn (2.6A), we have:

$$\mathcal{R}_{L,\mathbf{u}}(\mathbf{u}, \mathbf{y}; E) \le \mathrm{b}' \max_{\mathbf{v}\in\mathbb{Z}^{d\cdot N}\cap\partial^+\mathbf{B}_\ell(\mathbf{u})} \mathcal{R}_{L,\mathbf{u}}(\mathbf{v}, \mathbf{y}; E). \tag{2.9}$$

Here

$$\mathrm{b}' = \mathrm{c}'\, e^{-m\ell}\, |\partial\Lambda_\ell(u)| \ \le \mathrm{c}'\, e^{-m\ell}\ell^{d-1}, \tag{2.10}$$

where c is a constant from (2.3), (2.6.1,2), and $c' > 0$ is another 'geometric' constant (again depending only on dN).

2C. The LGRI for NR singular boxes. Now consider a situation where, for given $\mathbf{u} \in \mathbb{Z}^{d\cdot N}$, $E \in \mathbb{C}$, $m > 0$ and $\ell \in (0, L)$, box $\mathbf{\Lambda}_L(\mathbf{u})$ contains an (E, m)-S box $\mathbf{\Lambda}_\ell(\mathbf{v})$ with $\mathbf{v} \in \mathbf{B}_L(\mathbf{u})$, but

(i) any box $\mathbf{\Lambda}_\ell(\mathbf{v}') \subset \mathbf{\Lambda}_L(\mathbf{u})$ with $\mathbf{v}' \in \mathbf{B}_L(\mathbf{u})$ and with $\operatorname{dist}[\mathbf{\Lambda}_\ell(\mathbf{v}), \mathbf{\Lambda}_\ell(\mathbf{v}')] = 1$ (i.e., with $|\mathbf{v} - \mathbf{v}'| = 2\ell - 1$) is (E, m)-NS;

(ii) all boxes $\mathbf{\Lambda}_s(\mathbf{w}) \subset \mathbf{\Lambda}_L(\mathbf{u})$ with $\mathbf{w} \in \mathbf{B}_L(\mathbf{u})$ and $s \in [\ell, L]$ are E-NR.

In this situation the LGRI (2.6) implies that, $\forall$ $\mathbf{y} \in \mathbb{Z}^{d\cdot N} \cap \partial^-\mathbf{\Lambda}_L(\mathbf{u})$ and $\forall$ box $\mathbf{\Lambda}_\ell(\mathbf{v}) \subset \mathbf{\Lambda}_L(\mathbf{u})$ with $\mathbf{v} \in \mathbf{B}_L(\mathbf{u})$,

$$\begin{aligned}\mathcal{R}_{L,\mathbf{u}}(\mathbf{v}, \mathbf{y}; E) \le{}& \mathrm{c}e^{\ell^\beta} |\partial^+\mathbf{\Lambda}_\ell(\mathbf{v})|\\ &\times \max_{\mathbf{w}\in\mathbf{B}_L(\mathbf{u}):\ \mathbf{\Lambda}_\ell(\mathbf{w})\subset\mathbf{\Lambda}_L(\mathbf{u}),\ |\mathbf{w}-\mathbf{v}|=2\ell-1} \mathcal{R}_{L,\mathbf{u}}(\mathbf{w}, \mathbf{y}; E).\end{aligned} \tag{2.11}$$

Further, applying the LGRI to all neighboring boxes $\mathbf{\Lambda}_\ell(\mathbf{w})$, we arrive at the following bound:

$$\mathcal{R}_{L,\mathbf{u}}(\mathbf{v}, \mathbf{y}; E) \le \mathrm{b}'' \max_{\mathbf{w}\in\mathbf{B}_L(\mathbf{u}):\ \mathbf{\Lambda}_\ell\subset\mathbf{\Lambda}(\mathbf{u}),\ \ell\le|\mathbf{w}-\mathbf{v}|=2\ell-1} \mathcal{R}_{L,\mathbf{u}}(\mathbf{w}, \mathbf{y}; E) \tag{2.12}$$

with

$$\mathrm{b}'' = \mathrm{c}''\, e^{-m\ell} e^{\ell^\beta} \ell^{d-1}, \tag{2.13}$$

where $\mathrm{c}'' > 0$ is yet another 'geometric' constant. A helpful observation here is that all above mentioned boxes $\mathbf{\Lambda}_\ell(\mathbf{w})$ are contained in the 'layer'

$$\{\mathbf{x} :\ \ell \le \|\mathbf{x} - \mathbf{v}\| = 2\ell - 1\}$$

of width $2\ell - 1$ around box $\mathbf{\Lambda}_\ell(\mathbf{v})$.

More generally, given a number $\mathrm{A} \in (0, +\infty)$, suppose that a box $\mathbf{\Lambda}_\ell(\mathbf{v}) \subset \mathbf{\Lambda}_L(\mathbf{u})$ with $\mathbf{v} \in \mathbf{B}_L(\mathbf{u})$ is (E, m)-S, but:

(a) the box $\mathbf{\Lambda}_{A\ell}(\mathbf{v}) \subset \mathbf{\Lambda}_L(\mathbf{u})$ is E-NR;

(b) any box $\mathbf{\Lambda}_\ell(\mathbf{w}) \subset \mathbf{\Lambda}_L(\mathbf{u})$ such that $\mathbf{w} \in \mathbf{B}_L(\mathbf{u})$ and $\operatorname{dist}[\mathbf{\Lambda}_{A\ell}(\mathbf{v}), \mathbf{\Lambda}_\ell(\mathbf{w})] = 1$ is (E, m)-NS.

Then the analog of (2.12) reads as follows:

$$\mathcal{R}_{L,\mathbf{u}}(\mathbf{v},\mathbf{y};E) \leq \mathrm{b}''' \max_{\mathbf{w}\in\mathbf{B}_L(\mathbf{u}):\,\mathbf{\Lambda}\|\mathbf{w}-\mathbf{v}\|=(A+1)\ell-1} \mathcal{R}_{\mathbf{B}_L(\mathbf{u})}(\mathbf{w},\mathbf{y};E) \tag{2.14}$$

where

$$\mathrm{b}''' = \mathrm{c}''' e^{-m\ell} e^{\ell^\beta} \ell^{d-1},$$

where constant $\mathrm{c}''' > 0$ varies as $O(A^{d-1})$.

Observe that $\mathrm{b}' \leq \mathrm{b}''$, so that bound (2.9) implies a slightly weaker inequality

$$\mathcal{R}_{\Lambda_L(\mathbf{u})}(\mathbf{u},\mathbf{y};E)| \leq q \max_{\mathbf{v}\in\partial^+\Lambda_\ell(\mathbf{u})} \mathcal{R}_{\Lambda_L(\mathbf{u})}(\mathbf{v},\mathbf{y};E), \tag{2.15}$$

with the same value of q as in (2.12), (2.13). We see that the difference between bounds (2.9) and (2.12) resides in the form (and size) of the "reference set" of points $\mathbf{w}$ used in these recurrent relations.

2D. Clustering disjoint singular boxes. The remaining cases require an additional construction. Let us fix a box $\mathbf{\Lambda}_L(\mathbf{u})$ with $\mathbf{u}\in\mathbb{Z}^{d\cdot N}$ and suppose that $\mathbf{\Lambda}_L(\mathbf{u})$ contains some S-boxes of size 2ℓ with centers in $\mathbf{B}_L(\mathbf{u})$. In order to be able to apply to inequality (2.12) to a given S-box $\mathbf{\Lambda}_\ell(\mathbf{v}^{(1)}) \subset \mathbf{\Lambda}_L(\mathbf{u})$, with $\mathbf{v}^{(1)} \in \mathbf{B}_L(\mathbf{u})$, it is necessary to have all boxes of sidelength 2ℓ neighboring $\mathbf{\Lambda}_\ell(\mathbf{v}^{(1)})$, lying in $\mathbf{\Lambda}_L(\mathbf{u})$ and centred at a point from $\mathbf{B}_L(\mathbf{u})$, non-singular. However, it may happen that one of these neighbors, say $\mathbf{\Lambda}_\ell(\mathbf{v}^{(2)})$, is itself singular. In such a case we pass to a bigger box, $\mathbf{\Lambda}_{2\ell}(\mathbf{v}^{(1)}) \supset \mathbf{\Lambda}_\ell(\mathbf{v}^{(1)})$, and check for non-singularity its neighbors, $\mathbf{\Lambda}_\ell(\mathbf{v}^{(3)}) \subset \mathbf{\Lambda}_L(\mathbf{u}) \setminus \mathbf{\Lambda}_{2\ell}(\mathbf{v}^{(1)})$, with $\mathrm{dist}[\mathbf{\Lambda}_{2\ell}(\mathbf{v}^{(1)}), \mathbf{\Lambda}_\ell(\mathbf{v}^{(3)})] = 1$; again, one of these boxes can be singular. Then we pass to box $\mathbf{\Lambda}_{3\ell}(\mathbf{v}^{(1)})$ and repeat the checking procedure. Continuing, we obtain a finite sequence of singular sub-boxes of size 2ℓ which we will call a singular chain (an S-chain, for short):

$$\mathbf{\Lambda}_\ell(\mathbf{v}^{(1)}), \dots, \mathbf{\Lambda}_\ell(\mathbf{v}^{(n)}) \subset \mathbf{\Lambda}_L(\mathbf{u}), \ \ n \geq 1,$$

with

$$\mathrm{dist}[\mathbf{\Lambda}_{(k-1)\ell-1}(\mathbf{v}^{(1)}), \mathbf{\Lambda}_\ell(\mathbf{v}^{(k)}] = 1, \ \ k = 2, \dots, n.$$

Observe that, by construction, any two boxes in the above S-chain are disjoint. Moreover, in some situations we will need members of an S-chain positioned at a certain distance, viz.,

$$\mathrm{dist}[\mathbf{\Lambda}_\ell(\mathbf{v}^{(i)}), \mathbf{\Lambda}_\ell(\mathbf{v}^{(j)})] = b\ell, \ 1 \leq i \neq j \leq n.$$

Starting with one S-box, we can construct a maximal S-chain. It is not hard to see that if $\mathbf{\Lambda}_L(u)$ contains no singular chain with $> n$ elements, where $n \geq 1$, then for any point $\mathbf{x} \in \mathbf{B}_{L-2n\ell}(\mathbf{u})$ (i.e., for any point not too close to the boundary of the box $\mathbf{B}_L(\mathbf{u})$) the following inequality holds true:

$$\mathcal{R}_{L,\mathbf{u}}(\mathbf{v},\mathbf{y};E) \leq q \max_{\mathbf{w}:\|\mathbf{w}-\mathbf{v}\|=(A+1)\ell-1} \mathcal{R}_{\mathbf{B}_L(\mathbf{u})}(\mathbf{w},\mathbf{y};E),$$

with $A = A(\mathbf{v}) \leq 2n$.

In what follows, we call a maximal singular chain a singular cluster (briefly, an S-cluster).

It is worth mentioning that a box $\mathbf{\Lambda}_L(\mathbf{u})$ may contain, in principle, several S-clusters (i.e., several maximal S-chains); these S-clusters may contain different

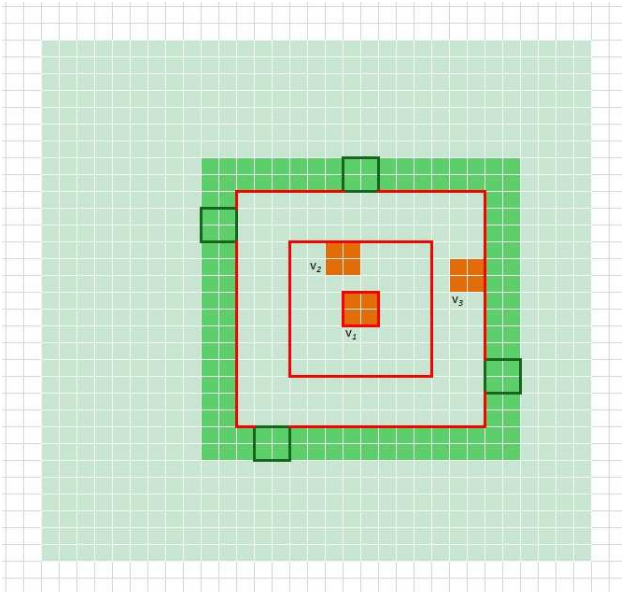

Fig. 1. A singular chain with 3 singular boxes (orange) centered at $\mathbf{v}^{(1)}$, $\mathbf{v}^{(2)}$, $\mathbf{v}^{(3)}$. Neighboring boxes inside the green annular area are NS (four of these NS-neighbors are singled out with dark green border)

numbers of members (disjoint S-boxes). For our purposes, it is not necessary to have S-clusters non-overlapping, although it is always possible, by properly making unions of S-boxes and surrounding such unions by larger boxes. This would produce what we call boxed singular clusters, or, briefly, BS-clusters. In fact, one can construct a finite number of non-overlapping BS-clusters such that

(i) no box of sidelength ℓ outside these BS-clusters is singular;

(ii) any box of sidelength ℓ adjacent to the boundary of a BS-cluster is NS;

(iii) if $\mathbf{\Lambda}_L(\mathbf{u})$ contains at most n non-overlapping S-boxes, then the sum of diameters of all BS-clusters occurring within $\mathbf{\Lambda}_L(\mathbf{u})$ is bounded by the product $C_7 n\ell$ where $C_7 = C_7(dN) \in (0, +\infty)$ is again a geometric constant.

In what follows we assume that S-clusters are constructed as described above, although, admittedly, such a construction is not unique. The most important property among (i)–(iii) is (iii): it asserts that all S-boxes $\mathbf{\Lambda}_\ell(\mathbf{v}) \subset \mathbf{\Lambda}_L(\mathbf{u})$ can be covered by a relatively small number of lattice boxes of size $O(n\ell)$, where n is the maximal number of disjoint S-boxes of size ℓ occurring in $\mathbf{\Lambda}_L(\mathbf{u})$.

2E. Subharmonicity of Green functions. Given a box $\mathbf{B}_L(\mathbf{u})$, fix $E \in \mathbb{R}$ and define a function $f : \mathbf{B}_L(\mathbf{u}) \to \mathbb{R}_+$ by

$$f(\mathbf{x}) = \max_{\mathbf{y} \in \partial^- \mathbf{B}_L(\mathbf{u})} \mathcal{R}_{\mathbf{B}_L(\mathbf{u})}(\mathbf{x}, \mathbf{y}; E). \tag{2.16}$$

Suppose that $\mathbf{B}_L(\mathbf{u})$ contains at least one S-cluster and define a set $\mathcal{S}$ as the union of all S-clusters. Then, by virtue of (2.11), for any lattice point $\mathbf{x} \notin \mathcal{S}$ we have

$$\mathcal{R}_{\mathbf{B}_L(\mathbf{u})}(\mathbf{u}, \mathbf{y}; E)| \le q \max_{\mathbf{v}:\, \|\mathbf{u}-\mathbf{v}\| = \ell - 1} \mathcal{R}_{\mathbf{B}_L(\mathbf{u})}(\mathbf{v}, \mathbf{y}; E), \tag{2.17}$$

while for points $\mathbf{x} \in \mathcal{S}$ we have, respectively,

$$\mathcal{R}_{\mathbf{B}_L(\mathbf{u})}(\mathbf{u}, \mathbf{y}; E)| \leq q \max_{\mathbf{v}:\, \ell \leq \|\mathbf{u}-\mathbf{v}\| = 2\ell - 1} \mathcal{R}_{\mathbf{B}_L(\mathbf{u})}(\mathbf{v}, \mathbf{y}; E), \tag{2.18}$$

with the same value of q. Obviously, if $\mathcal{S} = \varnothing$, then Eqn (2.17) can be used for all ℓ-boxes inside $\mathbf{B}_L(\mathbf{u})$, which only makes our estimates simpler.

In order to formalize such a property of a function f, we give the following

Definition 2.2. *Consider a box $\mathbf{B}_L(\mathbf{u})$ and a subset thereof $\mathcal{S} \subset \mathbf{B}_L(\mathbf{u})$. A function $f : \mathbf{B}_L(\mathbf{u}) \to \mathbb{R}_+$ is called $(q, \ell, \mathcal{S})$-subharmonic if for all points $\mathbf{x} \in \mathbf{B}_L(\mathbf{u}) \setminus \mathcal{S}$ with* $\operatorname{dist}[\mathbf{x}, \partial^- \mathbf{B}_L(\mathbf{u})] \geq \ell$ *we have*

$$f(\mathbf{x}) \leq q \max_{\mathbf{w}:\, \|\mathbf{w}-\mathbf{x}\| = 2\ell - 1} f(\mathbf{w}), \tag{2.19}$$

and for every point $\mathbf{x} \in \mathcal{S}$ there exists an integer $\rho(\mathbf{x}) \in [\ell, A\ell]$ and

$$f(\mathbf{x}) \leq q \max_{\mathbf{w}:\, \rho(\mathbf{x}) \leq \|\mathbf{w}-\mathbf{x}\| \leq \rho(\mathbf{x}) + 2\ell - 1} f(\mathbf{w}). \tag{2.20}$$

Remark. It is clear that, formally, we introduce the notion of $(\ell, q, \mathcal{S}, A)$-subharmonicity. The parameter A is dropped for notational simplicity only, and this should not lead to any ambiguity.

We see that under the above assumptions upon the box $\mathbf{B}_L(\mathbf{u})$, the function

$$f(\mathbf{x}) := \max_{\mathbf{y} \in \partial^- \mathbf{B}_L(\mathbf{u})} \mathcal{R}_{\mathbf{B}_L(\mathbf{u})}(\mathbf{x}, \mathbf{y}; E)$$

is $(q, \ell, \mathcal{S})$-subharmonic with $\mathcal{S}$ defined as a union of all singular clusters and

$$q = e^{-\gamma(m,\ell)} e^{\ell^\beta} C'(d) (n\ell)^{d-1}.$$

Moreover, it is not difficult to see that if any family of disjoint singular boxes

$$\mathbf{B}_\ell(\mathbf{v}^{(1)}), \mathbf{B}_\ell(\mathbf{v}^{(2)}), \ldots, \mathbf{B}_\ell(\mathbf{v}^{(j)}) \subset \mathbf{B}_L(\mathbf{u})$$

contains at most n elements, i.e. $j \leq n$, then the above function f is $(q, \ell, \mathcal{S})$-subharmonic with some set $\mathcal{S}$ (which is ***not*** defined in a unique way, in general) contained in a union of annular areas

$$\mathcal{A}(\mathcal{S}) := \bigcup_{i=1}^{j} \mathcal{A}_i, \ \mathcal{A}_i = \mathbf{B}_{b_i}(u) \setminus \mathbf{B}_{a_i}(u)$$

with $0 < a_1 < b_1 < a_2 \ldots < a_j < b_j < L$, $W(\mathcal{S}) := \sum_{i=1}^{j} (b_i - a_i) \leq 2n\ell$. We will call $W(\mathcal{S})$ the (total) width of the singular area $\mathcal{A}(\mathcal{S})$. If the annular covering $\mathcal{A}(\mathcal{S})$ is chosen in a minimal way, then $W(\mathcal{S})$ is uniquely defined.

In the next subsection, we will establish a general bound for subharmonic functions, making abstraction of exact values of parameter q.

2.1. Radial descent and decay of subharmonic functions. The following elementary statement is an adaptation of Lemma 4.3 from [C08]

Lemma 2.1. *[Radial Descent Lemma]*

$$f(\mathbf{u}) \leq q^{(L - W(\mathcal{S}) - 3\ell)/\ell} \mathcal{M}(f, \mathbf{B}_L(\mathbf{u})). \tag{2.21}$$

The proof can be found in [C08]; it is fairly straightforward.

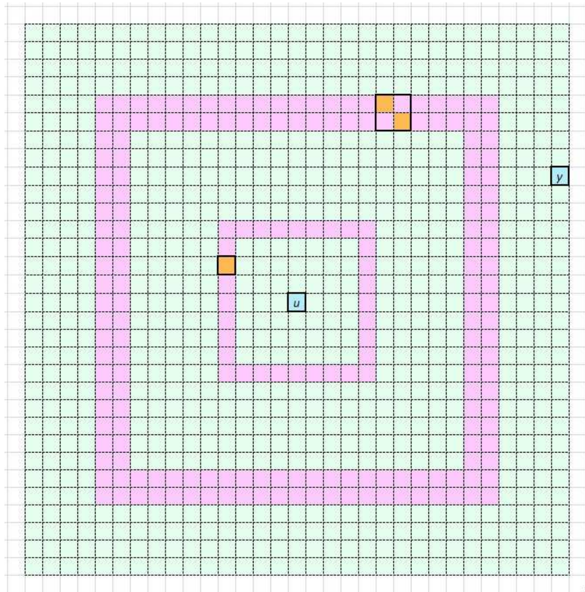

Fig. 2. An example of a box $\mathbf{B}_L(\mathbf{u})$ with two singular clusters (singular boxes are orange) covered by two annular areas (pink)

2.2. Application to the decay of Green functions. It is readily seen that Lemma 2.1 applied to the functions $f(\mathbf{u}) = \mathcal{R}(\mathbf{u}, \mathbf{v}; E)$ leads to the following

Lemma 2.2. *Fix a non-negative integer $n < \infty$ and suppose that a box $\mathbf{B}_L(\mathbf{u})$ is E-non-resonant and that any maximal family of b-distant (E, m)-singular boxes contains at most n elements. Then $\mathbf{B}_L(\mathbf{u})$ is (m, E)-non-singular:*

$$\max_{\mathbf{y} \in \partial^- \mathbf{B}_L(\mathbf{u})} |G_{\mathbf{B}_L(\mathbf{u})}(\mathbf{u}, \mathbf{y}; E)| \leq \exp\{ -\gamma(m, L) \}.$$

N.B.: It is clear from our above analysis that all arguments, as well as the statement of Lemma 2.2, remain valid for N-particle boxes in $\mathbb{R}^d$ (resp., two-particle boxes in $\mathbb{Z}^{d \cdot N}$. Indeed, apart from the difference in the value of the dimension and the additive structure of the potential $\mathbf{V}(x_1, x_2) = V(x_1) + V(x_2)$, the two-particle Hamiltonians similar form. Neither of these differences is crucial to our analysis, for the dimension can be arbitrary, and a particular structure of the potential is not used at all.

Note also that our analysis of $(\ell, q, \mathcal{S})$-subharmonic functions is purely "deterministic" and does not rely upon any probabilistic assumption relative to the random external potential $V(x; \omega)$.

This concludes our reduction of the deterministic part of the continuous MSA to the lattice version thereof. The rest of the proof of exponential decay of Green functions is conducted in terms of the auxiliary lattice model. The exponential decay of eigenfunctions is then deduced from that of Green functions in a standard way. A reader familiar with [CS09A] may notice that subsequent sections are straightforward adaptations of corresponding parts of [CS09A]; they do not contain truly novel ideas or techniques.

3. Partial decoupling and tunneling in two-particle boxes

Unlike the single-particle MSA, its two-particle counterpart proposed in [CS09A] has to address the following difficulty of multi-particle models: the probabilistic dependence between the values of the potential $\mathbf{V}(\mathbf{x};\omega) = V(x_1;\omega) + V(x_2;\omega)$ and $\mathbf{V}(\mathbf{y};\omega) = V(y_1;\omega) + V(y_2;\omega)$ does not decay with the distance $\|\mathbf{x} - \mathbf{y}\|$. However, a weaker form of "decoupling" in the potential $\mathbf{U}(\mathbf{x}) + \mathbf{V}(\mathbf{x};\omega)$ takes place for sufficiently distant points in the multi-particle configuration space. Such a decoupling, sufficient for the purposes of the two-particle MSA, makes use of the following elementary geometric statement (cf. [CS09A]):

Lemma 3.1. *Let be $L > r_0$ and consider two interactive boxes, $\mathbf{\Lambda}_L(\mathbf{u}')$ and $\mathbf{\Lambda}_L(\mathbf{u}'')$, with $dist(\mathbf{\Lambda}_L(\mathbf{u}'), \mathbf{\Lambda}_L(\mathbf{u}'')) > 8L$. Then*

$$\Pi\mathbf{\Lambda}_L(\mathbf{u}'') \cap \Pi\mathbf{\Lambda}_L(\mathbf{u}'') = \varnothing.$$

The proof is straightforward and can be found in [CS09A].

Further, for the purposes of estimates of probability of simultaneous (E,m)-singularity of two $8L$-distant boxes, making use of well-known results of the single-particle MSA, we introduce the following

Definition 3.1. *Given a bounded interval $I \subset \mathbb{R}$ and $m > 0$, a single-particle box $\Lambda_{L_k}(u)$ is called m-tunneling (m-T, for short) if $\exists\, E \in I$ and disjoint boxes $\Lambda_{L_{k-1}}(v_1)$, $\Lambda_{L_{k-1}}(v_2) \subset \Lambda_{L_k}(u)$ which are (E,m)-S. A two-particle box of the form $\mathbf{\Lambda}_{L_k}(\mathbf{u}) = \Lambda_{L_{k-1}}(u_1) \times \Lambda_{L_{k-1}}(u_2)$, with $\mathbf{u} = (u_1, u_2)$, is called m-tunneling (m-T) if either $\Lambda_{L_{k-1}}(u_1)$ or $\Lambda_{L_{k-1}}(u_2)$ is m-tunneling. Otherwise, it is called m-non-tunneling (m-NT, for short).*

It is worth mentioning that, while the notion of m-tunneling is, formally, defined for an arbitrary two-particle box, it is actually useful only in the case of a non-interactive box, where the spectral problem admits separation of variables, and so is reduced to two single-particle spectral problems.

The following statement is a reformulation of well-known results of the single-particle MSA (cf. [St01] and bibliography therein), so its proof is omitted.

Lemma 3.2. *Under the assumptions (E1–E4) upon the external (single-particle) external random potential $V(x;\omega)$,*

$$\mathbb{P}\{\, \Lambda_{L_k}(u) \text{ is } m\text{−T} \,\} \le L_k^{-q'}$$

where $q' = q'(\eta^*)$, $\eta^* := E_1^* - E_0^* > 0$, can be chosen so that $q'(\eta^*) \to +\infty$ as $\eta^* \downarrow 0$. Respectively, for a two-particle box $\mathbf{\Lambda}_{L_k}(\mathbf{u}) = \Lambda_{L_{k-1}}(u_1) \times \Lambda_{L_{k-1}}(u_2)$ we have

$$\mathbb{P}\{\, \mathbf{\Lambda}_{L_k}(u) \text{ is } m\text{−T} \,\} \le \sum_{j=1}^{2} \mathbb{P}\{\, \Lambda_{L_k}(u_j) \text{ is } m\text{−T} \,\} \le 2L_k^{-q'}.$$

4. Reduction of the localization problem to the MSA

Theorem 4.1. *Suppose that for some $m > 0$ and all $k \geq 0$ the following bound holds true: for any pair of L_k-distant two-particle boxes $\mathbf{B}_{L_k}(\mathbf{u}')$ and $\mathbf{B}_{L_k}(\mathbf{u}'')$,*

$$\mathbb{P}\left\{ \exists\, E \in [E_0^*, E_1^*] : \mathbf{B}_{L_k}(\mathbf{u}') \text{ and } \mathbf{B}_{L_k}(\mathbf{u}'') \text{ are } (E,m)\text{-}S \right\} \leq L_k^{-2p}. \tag{4.1}$$

Then with probability one, the spectrum of operator $\mathbf{H}(\omega)$ in $[E_0^*, E_1^*]$ is pure point, and for any EF $\mathbf{\Psi}_j(\mathbf{x};\omega)$ with $E_j(\omega) \in [E_0^*, E_1^*]$, we have, for any $\mathbf{v} \in \mathbb{Z}^{d\cdot N}$:

$$\|\mathbf{1}_{\mathbf{C}(\mathbf{v})}\, \mathbf{\Psi}_j(\cdot;\omega)\| \leq C_j(\omega) e^{-m\|\mathbf{v}\|}. \tag{4.2}$$

For the reader's convenience, we give the proof of the above theorem in Section 4. All its ingredients can be found in [CS09A] (as far as the two-particle structure of the Hamiltonian is concerned)) and in [St01].

Therefore, Anderson localization will be established, once we prove the main probabilistic bound of the MSA given by Eqn (4.1).

As usual in the MSA, the probabilistic bound (4.1) is first established for $k = 0$ initial length scale estimates), and then proved inductively for all $k \geq 1$.

The proof of the initial length scale estimate is completely analogous to that in the conventional, single-particle localization theory, and is omitted for this reason. Indeed, the reader may check that the arguments used, e.g., in [St01] (cf. Ch. 3.3, pp. 90–98) do not use any assumption on the structure of the external potential which is not satisfied in the two-particle (actually, even N-particle, with $N \geq 1$) model. The basis for these initial scale estimates is the well-known Combes-Thomas bound (cf. [CT73]), combined with the fact that we consider energies $E \in [E_0^*, E_1^*]$ sufficiently close to the lower edge E_0^* of the spectrum.

So, in the rest of the paper, we focus on the inductive proof of the bound (4.1). To this end, we consider two kinds of boxes:

(i) *non-interactive* boxes $\mathbf{B}_L(\mathbf{u}) = \mathbf{\Lambda}_L(\mathbf{u}) \cap \mathbb{Z}^{d\cdot N}$ where the interaction potential vanishes: $\mathbf{U}|_{\mathbf{\Lambda}_L(\mathbf{u})} \equiv 0$;

(ii) *interactive* boxes $\mathbf{B}_L(\mathbf{u}) = \mathbf{\Lambda}_L(\mathbf{u}) \cap \mathbb{Z}^{d\cdot N}$ where the interaction potential is not identically zero on $\mathbf{\Lambda}_L(\mathbf{u})$.

This gives rise to three categories of *pairs* of (sufficiently distant) boxes:

(I) Two non-interactive boxes.

(II) Two interactive boxes.

(III) A mixed pair of one interactive and one non-interactive box.

These three cases will be treated separately in sections 5, 6 and 7, respectively.

By virtue of Theorem 4.1, Anderson localization (cf. Theorem 1) will be proven for the two-particle system in $\mathbb{R}^d$ with an alloy-type external random potential, verifying conditions (D), (E1)-(E4) given in Section 1, once the bound (4.1) is established in all cases (I)-(III).

Remark. For the sake of notational simplicity, below we will call a box $\mathbf{B}_L(\mathbf{u}) = \mathbf{\Lambda}_L(\mathbf{u}) \cap \mathbb{Z}^{d\cdot N}$ E-non-resonant (resp., E-non-resonant) iff the corresponding box $\mathbf{\Lambda}_L(\mathbf{u}) \subset \mathbb{R}^{d\cdot N}$ is E-resonant (resp., E-non-resonant).

5. Pairs of non-interactive boxes

We begin with an auxiliary result about non-interactive boxes, which was earlier used in [CS09A], [CS09B]. For the reader's convenience, we give its proof (which is straightforward) in the Appendix.

Lemma 5.1. *Suppose that a two-particle box* $\mathbf{B}_{L_{k+1}}(\mathbf{u})$ *is* E*-non-resonant and satisfies the following property: for any pair of sub-boxes* $\mathbf{B}_{L_k}(\mathbf{v}'), \mathbf{B}_{L_k}(\mathbf{v}') \subset \mathbf{B}_{L_{k+1}}(\mathbf{u})$ *with* $\mathrm{dist}[\mathbf{B}_{L_k}(\mathbf{v}'), \mathbf{B}_{L_k}(\mathbf{v}'')] > 8L_k$*, either* $\mathbf{B}_{L_k}(\mathbf{v}')$ *or* $\mathbf{B}_{L_k}(\mathbf{v}')$ *is* (E,m)*-non-singular. Then* $\mathbf{B}_{L_{k+1}}(\mathbf{u})$ *is also* (E,m)*-non-singular.*

Proof of (4.1) for a pair of non-interactive boxes.

Consider a pair of two-particle non-interactive boxes $\mathbf{B}' = \mathbf{B}_{L_{k+1}}(\mathbf{u}')$, $\mathbf{B}'' = \mathbf{B}_{L_{k+1}}(\mathbf{u}'')$, and introduce the events

$$\begin{aligned} \mathbf{T} &= \{ \text{either } \Lambda' \text{ or } \Lambda'' \text{ is } m\text{-T } \}, \\ \mathbf{R} &= \{ \exists\, E \in [E_0, E_1] : \text{ both } \Lambda' \text{ and } \Lambda'' \text{ are } E\text{-R } \}, \\ \mathbf{S} &= \{ \exists\, E \in [E_0, E_1] : \text{ both } \Lambda' \text{ and } \Lambda'' \text{ are } (E,m)\text{-S } \}. \end{aligned}$$

Then we can write

$$\mathbb{P}\{\, \mathbf{S} \,\} \le \mathbb{P}\{\, \mathbf{T} \,\} + \mathbb{P}\{\, \mathbf{S} \cap \mathbf{T}^c \,\}.$$

Owing to Lemma 3.2, we have

$$\mathbb{P}\{\, \mathbf{T} \,\} \le \mathbb{P}\{\, \Lambda' \text{ is } m\text{-T } \} + \mathbb{P}\{\, \Lambda'' \text{ is } m\text{-T } \} \le 2\, 2L_k^{-q'},$$

where $q' > 0$ can be chosen arbitrarily large, provided that $E_1^* - E_0^*$ is sufficiently small. So, we can pick $q' \ge q$ with $q > 0$ given in the Wegner-type bound (W2). Further, by two-volume Wegner-type estimate (W2), we have

$$\mathbb{P}\{\, \mathbf{R} \,\} < L_k^{-q}.$$

By virtue of the NITRoNS (Lemma 5.1), $\mathbf{S} \cap \mathbf{T}^c \subset \mathbf{R}$, Now, using (W2), we obtain

$$\mathbb{P}\{\, \mathbf{S} \,\} \le 4L_k^{-q'} + \mathbb{P}\{\, \mathbf{R} \,\} \le L_k^{-q'} + L_k^{-q} \le 2L_k^{-q} < L_k^{-2p},$$

owing to our choice of parameter $q (> 3p + 9)$, for all sufficiently large L_k. Thus, the bound (4.1) is proven for distant pairs of non-interactive boxes.

6. Pairs of interactive boxes

Consider again the following events:

$$\begin{aligned} \mathbf{R} &= \{ \exists\, E \in [E_0, E_1] : \text{ both } \mathbf{B}' \text{ and } \mathbf{B}'' \text{ are } E\text{-R } \}, \\ \mathbf{S} &= \{ \exists\, E \in [E_0, E_1] : \text{ both } \mathbf{B}' \text{ and } \mathbf{B}'' \text{ are } (E,m)\text{-S } \}. \end{aligned}$$

Using the Wegner-type bound (W2) and our condition $q > 3p + 9$, we see that

$$\mathbb{P}\{\, \mathbf{S} \,\} \le \mathbb{P}\{\, \mathbf{R} \,\} + \mathbb{P}\{\, \mathbf{S} \cap \mathbf{R}^c \,\} \le \frac{1}{2} L_k^{-2p} + \mathbb{P}\{\, \mathbf{S} \cap \mathbf{R}^c \,\}. \tag{6.1}$$

Within the event $\mathbf{R}^c$, either $\mathbf{B}'$ or $\mathbf{B}''$ is E-non-resonant. Without loss of generality, assume that $\mathbf{B}'$ is E-non-resonant.

By virtue of the Radial Descent Lemma, if $\mathbf{B}'$ is (E,m)-singular, but E-non-resonant, then it must contain a singular cluster of $2M+1 \geq 5$ (with $M=2$) distant sub-boxes $\mathbf{B}_{L_k}(\mathbf{u}_j)$, $j=1,\dots,2M+1$.

Consider the following events:

$$\begin{aligned}
\mathbf{S}'_I &= \{\, \mathbf{B}' \text{ contains at least two } (E,m)\text{-S non-interactive boxes}\,\},\\
\mathbf{S}'_{NI} &= \{\, \mathbf{B}' \text{ contains at least } 2M \geq 4 \ (E,m)\text{-S interactive boxes}\,\}.
\end{aligned}$$

Obviously, $\mathbf{S}' \subset \mathbf{S}'_I \cup \mathbf{S}'_{NI}$.

Reasoning as in Section 5, we conclude that $\mathbb{P}\{\, \mathbf{S}'_I \,\} \leq 2L_k^{-q}$.

Further, suppose that $\mathbf{B}'$ contains at least $2M$ (E,m)-singular distant interactive boxes $\mathbf{B}_{L_k}(\mathbf{u}_j)$, $j=1,\dots,2m$. Owing to Lemma 3.1, the external potential samples in boxes $\mathbf{B}_{L_k}(\mathbf{u}_j)$ are independent. The situation here is completely analogous to that in the single-particle theory, and we can write that

$$\begin{aligned}
&\mathbb{P}\{\, \mathbf{S}'_{NI} \,\}\\
&\leq L_{k+1}^{2M(d+\alpha^{-1})} \prod_{i=1}^{M} \mathbb{P}\{\, \exists\, E \in [E_0,E_1]:\ \mathbf{B}_{L_k}(\mathbf{u}_j) \text{ and } \mathbf{B}_{L_k}(\mathbf{u}_j) \text{ are } (E,m)\text{-S}\,\}\\
&\leq L_{k+1}^{2M(d+\alpha^{-1})} \left(L_k^{-2p}\right)^M < \tfrac{1}{2} L_{k+1}^{-2p},
\end{aligned}$$

as long as $p > \frac{3d}{2}+1$, with $M=2$, and L_0 (hence, every L_k, $k\geq 1$) is sufficiently large. Taking into account Eqn (6.1), we see that

$$\mathbb{P}\{\, \mathbf{S} \,\} \leq \frac{1}{2} L_{k+1}^{-2p} + \frac{1}{2} L_{k+1}^{-2p} = L_{k+1}^{-2p},$$

yielding the bound (4.1) for pairs of (distant) interactive boxes.

7. Mixed pairs of boxes

It remains to derive the bound (4.1) in case (III), i.e., for mixed pairs of two-particle boxes: an interactive box $\mathbf{B}_{L_{k+1}}(\mathbf{x})$ and a non-interactive box $\mathbf{B}_{L_{k+1}}(\mathbf{y})$. Here we use several properties which have been established earlier in this paper for all scale lengths, namely, **(W1)**, **(W2)**, **NITRoNS**, and the bound (4.1) for pairs of (distant) non-interactive boxes, in Section 5.

Consider the following events:

$$\begin{aligned}
\mathbf{S} &= \left\{ \exists\, E \in I:\ \text{both } \mathbf{B}_{L_{k+1}}(\mathbf{x}),\ \ \mathbf{B}_{L_{k+1}}(\mathbf{y}) \ \text{ are } (E,m)\text{-S} \right\},\\
\mathbf{T} &= \left\{ \ \mathbf{B}_{L_{k+1}}(\mathbf{y}) \text{ is } m_0\text{-T} \right\},\\
\mathbf{R} &= \left\{ \exists\, E \in I:\ \ \text{neither } \mathbf{B}_{L_{k+1}}(\mathbf{x}) \text{ nor } \mathbf{B}_{L_{k+1}}(\mathbf{y}) \text{ is } (E,J)\text{-NR} \right\}.
\end{aligned}$$

As before, we have

$$\mathbb{P}\{\, \mathbf{T} \,\} \leq L_{k+1}^{-q'} \leq L_{k+1}^{-q}, \quad \mathbb{P}\{\, \mathbf{R} \,\} \leq L_{k+1}^{-q}. \tag{5.3}$$

Further,

$$\mathbb{P}\{\, \mathbf{S} \,\} \leq \mathbb{P}\{\, \mathbf{T} \,\} + \mathbb{P}\{\, \mathbf{S} \cap \mathbf{T}^{\mathrm{c}} \,\} \leq \frac{1}{4} L_{k+1}^{-2p} + \mathbb{P}\{\, \mathbf{S} \cap \mathbf{T}^{\mathrm{c}} \,\},$$

and for the last term in the RHS we have

$$\mathbb{P}\{\mathbf{S}\cap\mathbf{T}^{c}\} \le \mathbb{P}\{\mathbf{R}\} + \mathbb{P}\{\mathbf{S}\cap\mathbf{T}^{c}\cap\mathbf{R}^{c}\} \le L_{k+1}^{-q+2} + \mathbb{P}\{\mathbf{S}\cap\mathbf{T}^{c}\cap\mathbf{R}^{c}\}.$$

Within the event $\mathbf{S}\cap\mathbf{T}^{c}\cap\mathbf{R}^{c}$, either $\mathbf{B}_{L_{k+1}}(\mathbf{x})$ or $\mathbf{B}_{L_{k+1}}(\mathbf{y})$ is E-non-resonant. It must be the interactive box $\mathbf{B}_{L_{k+1}}(\mathbf{x})$. Indeed, by **NITRoNS**, had box $\mathbf{B}_{L_{k+1}}(\mathbf{y})$ been both E-non-resonant and m-non-tunneling, it would have been (E,m)-non-singular, which is not allowed within the event $\mathbf{S}$. So, the box $\mathbf{B}_{L_{k+1}}(\mathbf{x})$ must be E-non-resonant, but (E,m)-singular:

$$\mathbf{S}\cap\mathbf{T}^{c}\cap\mathbf{R}^{c} \subset \{\exists\, E\in I:\ \mathbf{B}_{L_{k+1}}(\mathbf{x}) \text{ is } (E,m)\text{-S and } E\text{-NR}\}.$$

However, applying the Radial Descent Lemma, we see that

$$\begin{aligned}&\{\exists\, E\in I:\ \mathbf{B}_{L_{k+1}}(\mathbf{x}) \text{ is } (E,m)\text{-S and } E\text{-NR}\}\\ &\qquad \subset \{\exists\, E\in I:\ K(\mathbf{B}_{L_{k+1}}(\mathbf{x});E)\ge J+1\}.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathbb{P}\{\mathbf{S}\cap\mathbf{T}^{c}\cap\mathbf{R}^{c}\} &\le \mathbb{P}\left\{\exists\, E\in I:\ K(\mathbf{B}_{L_{k+1}}(\mathbf{x});E)\ge 2\ell+2\right\}\\ &\le 2L_{k+1}^{-1}\,L_{k+1}^{-2p}.\end{aligned}$$

Finally, we get, with $q'' := q/\alpha = 2q/3 > 2p+6$,

$$\begin{aligned}\mathbb{P}\{\mathbf{S}\} &\le \mathbb{P}\{\mathbf{S}\cap\mathbf{T}\} + \mathbb{P}\{\mathbf{R}\} + \mathbb{P}\{\mathbf{S}\cap\mathbf{T}^{c}\cap\mathbf{R}^{c}\}\\ &\le \tfrac{1}{2}L_{k+1}^{-2p} + L_{k+1}^{-2p-2} + 2L_{k+1}^{-1}\,L_{k+1}^{-2p} \le L_{k+1}^{-2p},\end{aligned}$$

This completes the proof of bound (4.1).

Therefore, Theorem 1 is also proven and the Anderson localization established for a two-particle model satisfying hypotheses ***(D)*** *and* ***(E1)*** *–* ***(E4)****.*

8. Appendix. Proof of NITRoNS principle

Here we give the proof of Lemma 5.1. Recall that we consider operator $\mathbf{H}^{\mathbf{\Lambda}_{L_k}(\mathbf{u})}$ in a box $\mathbf{\Lambda}_{L_k}(\mathbf{u})$ and "single-particle" operators $H^{\Lambda^{L_k}(u')}$, $H^{\Lambda^{L_k}(u'')}$. Let $\{\varphi_a,\lambda_a\}$ be normalized eigenfunctions and the respective eigenvalues of $H^{\Lambda^{L_k}(u')}$. Similarly, let $\{\psi_b,\mu_b\}$ be normalized eigenfunctions and the respective eigenvalues of $H^{\Lambda^{L_k}(u'')}$.

Consider the Green functions $\mathbf{G}(\mathbf{v},\mathbf{y};E_j) \equiv \mathbf{G}^{\mathbf{\Lambda}_{L_k}(\mathbf{u})}(\mathbf{v},\mathbf{y};E_j)$, $\mathbf{v},\mathbf{y} \in \mathbf{B}_{L_k}(\mathbf{u})$. Observe that, since the external potential is non-negative, so are the eigenvalues $\{\lambda_a\}$ and $\{\mu_b\}$. Therefore, if $E\le E_1^*$, then we also have $E-\lambda_a\le E_1^*$, $E-\mu_b\le E_1^*$, for all λ_a and μ_b.

By the hypothesis of the lemma, $\mathbf{\Lambda}_{L_k}(\mathbf{u})$ is E-non-resonant. Therefore, for all λ_a, the 1-particle box $\Lambda_{L_k}(u'')$ is $(E-\lambda_a)$-non-resonant. By the assumption of m-non-tunneling, $\forall\, E\in[E_0^*,E_1^*]$ box $\Lambda^{L_k}(u'')$ must not contain two disjoint $(E-\lambda_a,m)$-singular sub-boxes of size L_{k-1}. Therefore, the Radial Descent Lemma implies that $\Lambda_{L_k}(u'')$ is $(E-\lambda_a)$-non-singular, yielding the required upper bound.

Let us now prove the second assertion of the lemma. If $\mathbf{u} = (u', u'')$ and $\mathbf{v} = (v', v'') \in \partial \mathbf{\Lambda}_{L_k}(\mathbf{u})$, then either $\|u' - v'\| = L_k$, or $\|u'' - v''\| = L_k$. In the former case we can write

$$\begin{aligned}\mathbf{G}(\mathbf{u}, \mathbf{v}; E) &= \sum_a \varphi_a(u')\varphi_a(v') \sum_b \frac{\psi_b(u'')\psi_b(v'')}{(E - \lambda_a) - \mu_b} \\ &= \sum_a \varphi_a(u')\varphi_a(v') \; G^{\Lambda_{L_k}(u'')}(u'', v''; E - \lambda_a).\end{aligned} \tag{9.1}$$

As mentioned above, $E - \lambda_a \le E_1^*$. In fact, by Weyl's law, $E - \lambda_a \to -\infty$ as $a \to \infty$. More precisely, for all $a \ge a^* = C^* |\Lambda_{L_k}(u')|$ (with constant C^* given by the Weyl's law), we have $E - \lambda_a \le -m^*$, where $m^* > 0$ can be chosen arbitrarily large, and, therefore, $E - \lambda_a < 0$ is far away from the (positive) spectrum:

$$\mathrm{dist}[\mathbf{\Sigma}\ (\mathrm{H}^{\Lambda_{\mathrm{L_k}}(\mathrm{u}')}), \mathrm{E} - \lambda_{\mathrm{a}}] = |\mathrm{E}_0 - (\mathrm{E} - \lambda_{\mathrm{a}})| \ge \mathrm{m}^*.$$

By virtue of the Combes–Thomas estimate, if $E - \lambda_a \le -m^*$ and $m^* > 0$ is large enough, then

$$\max_{v' \in \partial \Lambda_{L_k}(u')} \|\mathbf{1}_{C(u'')} G^{\Lambda_{L_k}(u'')}(E - \lambda_a) \mathbf{1}_{C(v'')}\| \le e^{-m\|u' - u''\|} \le e^{-mL_k}.$$

On the other hand, given any non-negative number m^*, one can consider from the beginning the energy interval $[-m^*, E_1^*]$ instead of $[E_0^*, E_1^*]$. Considering negative energies is fictitious, yet the standard, single-particle MSA would imply, formally, all required probabilistic MSA estimates for such a larger interval $[-m^*, E_1^*]$. The same is true, of course, for the two-particle MSA.

Therefore, an infinite sum over a in (9.1) can be divided into two sums:

$$\mathbf{G}(\mathbf{u}, \mathbf{v}; E) = \left(\sum_{a \le a^*} + \sum_{a > a^*} \right) \varphi_a(u')\varphi_a(v') \; G^{\Lambda_{L_k}(u'')}(u'', v''; E - \lambda_a), \tag{9.1'}$$

where the (infinite) sum $\sum_{a > a^*}(\cdot)$ can be made smaller than, for example, e^{-2mL_k}, by choosing a^* large enough, thus making $m^* > 0$ large enough. On the other hand, the first sum, $\sum_{a \le a^*}(\cdot)$, contains a finite number of terms: $O(L_k^d)$.

Since $\|\varphi_a\| = 1$ for all a, we see that

$$\begin{aligned}&\|\mathbf{1}_{\mathbf{C}(\mathbf{u})} \mathbf{G}(E) \mathbf{1}_{\mathbf{C}(\mathbf{v})}\| \\ &\le e^{-2mL_k} + C' \, |\Lambda_{L_k}(u')| \, \max_{a \le a^*} \|\mathbf{1}_{C(u'')} G^{\Lambda_{L_k}(u'')}(E - \lambda_a) \mathbf{1}_{C(v'')}\| \\ &\le C'' \, (2L_k)^d \, e^{-mL_k},\end{aligned} \tag{9.2}$$

owing to the $(E - \lambda_a, m)$-non-singularity of the box $\Lambda_{L_k}(u'')$.

In the case where $\|u'' - v''\| = L_k$, we can use the representation

$$\mathbf{G}(\mathbf{u}, \mathbf{v}; E) = \sum_b \psi_b(u'')\psi_b(v'') \; G^{\Lambda_{L_k}(u')}(u', v'; E - \mu_b). \tag{9.3}$$

□

9. Appendix B. Proof of the Theorem 4.1

Lemma 9.1. *Fix an interval* $I = [E_0, E_1] \subset \mathbb{R}$ *and a sequence of positive numbers* $\{L_k = (L_0)^{\alpha^k}\}$, $L_0 > 0$, $\alpha \in (1,2)$. *Suppose that the bounds (???) are satisfied for all* $\kappa \geq 0$.

Then there exists a positive number m *and a subset* $\Omega_0 \subset \Omega$ *with* $\mathbb{P}\{\,\Omega_0\,\} = 1$ *such that for every* $E \in I$ *and* $\omega \in \Omega_0$ *and for every polynomially bounded function* $\mathbf{f} \in \mathcal{L}_{2,\mathrm{loc}}(\mathbb{R}^{d\cdot N})$ *satisfying*

$$\|\mathbf{1}_{\mathbf{C}(\mathbf{v})}\mathbf{f}\| \leq C(\mathbf{f}) \cdot \|\chi^{\mathrm{out}}_{\ell,\mathbf{v}}\, \mathcal{R}_{\ell,\mathbf{v}}(E;\omega)\, \chi^{\mathrm{int}}_{\ell,\mathbf{v}}\| \cdot \|\chi^{\mathrm{out}}_{\ell,\mathbf{v}}\, \mathbf{f}\| \tag{9.1}$$

there exists $C = C(\mathbf{f}, \omega, m)$ *such that*

$$\|\mathbf{1}_{\mathbf{C}(\mathbf{v})}\mathbf{f}\| \leq C\, e^{-m\|\mathbf{v}\|} \tag{9.2}$$

and, more precisely,

$$\limsup_{\|\mathbf{v}\|\to\infty} \frac{\ln(\|\mathbf{1}_{\mathbf{C}(\mathbf{v})}\mathbf{f}\|)}{\|\mathbf{v}\|} \leq -m. \tag{9.3}$$

Proof. Let $R : \mathbb{R}^{d\cdot N} \to \mathbb{R}_+$ be the function given by $R(\mathbf{u}) = \|\mathbf{u} - S(\mathbf{u})\|$, where $S(u_1, u_2) = (u_2, u_1)$, $\mathbf{u} = (u_1, u_2) \in \mathbb{R}^{d\cdot N}$. Next, for every $k \in \mathbb{N}$, set

$$b_k(\mathbf{u}) = 1 + R(\mathbf{u})L_k^{-1},\ \mathbf{M}_k(\mathbf{u}) = \mathbf{\Lambda}_{L_k}(\mathbf{u}) \cup S\left(\mathbf{\Lambda}_{L_k}(\mathbf{u})\right).$$

Observe that for any $\mathbf{u} \in \mathbb{R}^{d\cdot N}$ we have

$$\forall k \geq 0:\ \mathbf{M}_k(\mathbf{u}) \subset \mathbf{\Lambda}_{b_{k+1}L_k}(\mathbf{u}), \qquad \text{and} \lim_{k\to\infty} b_k(\mathbf{u}) = 1. \tag{9.4}$$

Next, introduce annular subsets of the lattice

$$A_{k+1}(\mathbf{x}_0) = \mathbf{\Lambda}_{2bL_{k+1}}(\mathbf{x}_0) \setminus \mathbf{\Lambda}_{2L_k}(\mathbf{x}_0) \cap \mathbb{Z}^{d\cdot N}$$

centered at points $\mathbf{x}_0 \in \mathbb{Z}^{d\cdot N} \subset \mathbb{R}^{d\cdot N}$. Next, consider events

$$\mathbf{S}_k(\mathbf{u}) = \{\exists\, E \in I, \mathbf{x} \in A_{k+1}(\mathbf{x}_0):\ \mathbf{\Lambda}_{L_k}(\mathbf{u}) \text{ and } \mathbf{\Lambda}_{L_k}(\mathbf{x}) \text{ are } (E,m)\text{–S}\}.$$

Observe that, owing to the definition of $\mathbf{M}_k(\mathbf{u})$, if $\mathbf{x} \in \mathbf{A}_{k+1}(\mathbf{u})$, then

$$\mathrm{dist}[\mathbf{\Lambda}_k(\mathbf{u}), \mathbf{M}_k(\mathbf{u})] > 8L_k, \tag{9.5}$$

and, by the hypothesis of the lemma,

$$\mathbb{P}\{\,\mathbf{S}_k(\mathbf{u})\,\} \leq \frac{(2b_{k+1}L_{k+1}+1)^2 d}{L_k^{2p}} \leq \frac{(2b_{k+1}+1)^2 d}{L_k^{2p-2\alpha}}. \tag{9.6}$$

Since $p > \alpha$, and by virtue of (9.4), $\sum_{k\geq 0} \mathbb{P}\{\,\mathbf{S}_k(\mathbf{u})\,\} < \infty$, and the event

$$\mathbf{S}_\infty(\mathbf{u}) = \{\,\mathbf{S}_k(\mathbf{u}) \text{ occurs infinitely many times }\,\}$$

has probability zero, by virtue of the Borel–Cantelli lemma. As a consequence, the event

$$\mathbf{S}_\infty = \bigcup_{\mathbf{u}\in\mathbb{Z}^{d\cdot N}} \mathbf{S}_\infty(\mathbf{u})$$

also has probability zero, so that its complement

$$\Omega_0 = \{\forall\, \mathbf{v} \in \mathbb{Z}^{d\cdot N}\ \exists\, k_{\mathbf{v}}(\omega) \in \mathbb{N} \ \text{ such that } \forall\, k \geq k_{\mathbf{v}}(\omega)\ \mathbf{S}_k(\mathbf{v}) \not\ni \omega\}.$$

has probability 1.

The rest of the proof is purely "deterministic". Let $E \in I$, $\omega \in \Omega_0$ and $\mathbf{f} \in \mathcal{L}_{2,\mathrm{loc}}(\mathbb{R}^{d\cdot N})$ a polynomially bounded function satisfying Eqn. (9.1). If $\mathbf{f} \neq 0$, then there exists a lattice point $\mathbf{x}_0$ such that $\|\mathbf{1}_{\mathbf{C}(\mathbf{x}_0)}\mathbf{f}\| > 0$; we pick such a point $\mathbf{x}_0$ and fix it for the rest of the proof. The box $\mathbf{\Lambda}_{L_k}(\mathbf{x}_0)$ cannot be (E,m)-nonsingular for infinitely many values of k, since it would imply that

$$\|\mathbf{1}_{\mathbf{C}(\mathbf{x}_0)}\mathbf{f}\| \leq Const\, L_k^{2d-1} e^{-mL_k} \underset{k\to\infty}{\longrightarrow} 0, \tag{9.7}$$

hence, $\|\mathbf{1}_{\mathbf{C}(\mathbf{x}_0)}\mathbf{f}\| = 0$, in contradiction with our hypothesis. Thus, there exists some k_0 such that $\forall\, k \geq k_0$ the box $\mathbf{\Lambda}_{L_k}(\mathbf{x}_0)$ cannot be (E,m)-singular. In turn, this means, by construction of the event A_{k+1}, that for any point $\mathbf{x} \in A_{k+1}(\mathbf{x}_0)$ the box $\mathbf{\Lambda}_{L_k}(\mathbf{x}_0)$ is (E,m)-nonsingular.

Further, set

$$\mathbf{B}_{k+1}(\mathbf{x}_0) = \mathbf{\Lambda}_{\frac{2b}{1+\rho}L_{k+1}}(\mathbf{x}_0) \setminus \mathbf{\Lambda}_{\frac{2}{1-\rho}L_k}(\mathbf{x}_0) \subset A_{k+1}(\mathbf{x}_0).$$

It is readily seen that for any $\mathbf{x} \in \mathbf{B}_{k+1}$, we have $\mathrm{dist}[x, \mathbf{B}_{k+1}(\mathbf{x}_0)] \geq \rho\, \|\mathbf{x} - \mathbf{x}_0\|$. Furthermore, if $\|\mathbf{x} - \mathbf{u}\| \geq L_0/(1-\rho)$, then $\exists\, k \geq 0$ such that $\mathbf{x} \in \mathbf{B}_{k+1}(\mathbf{u})$.

Now we see that for sufficiently big $k \geq 0$, the box $\mathbf{\Lambda}_{L_k}(\mathbf{u})$ is (E,m)-nonsingular, so that $E \notin \mathrm{spec}\,(\mathbf{H}^{\mathbf{\Lambda}_{L_k}(\mathbf{u})})$. Therefore, we can apply the GRI and obtain

$$\|\mathbf{1}_{C_1(\mathbf{x})}\mathbf{f}\| \leq C(d) L_k^{2d-1} e^{-mL_k} \max_{\mathbf{v}\ldots} \|\mathbf{1}_{C_1(\mathbf{v})}\mathbf{f}\| \tag{9.8}$$

Pick a value $\widetilde{\rho} \in (0,1)$ and write it as a product of the form $\widetilde{\rho} = \rho\rho'$ with some $\rho, \rho' \in (0,1)$. Pick also a number $b > 8 + 1 + \rho/(1-\rho)$. The above inequality (9.8) can iterated at least $n_k := ((L_k+1)^{-1}\rho\|\mathbf{x} - \mathbf{u}\|$ times, producing the following bound:

$$\|\mathbf{1}_{C_1(\mathbf{x})}\mathbf{f}\| \leq \left(C(d) L_k^{2d-1} e^{-mL_k}\right)^{n_k} Const\,(1 + \|\mathbf{u}\| + bL_{k+1})^t.$$

Therefore, for k big enough, if $\|\mathbf{x} - \mathbf{u}\| \geq L_k/(1-\rho)$, then

$$\|\mathbf{1}_{C_1(\mathbf{x})}\mathbf{f}\| \leq e^{-\rho\rho' m \|\mathbf{x}-\mathbf{u}\|},$$

so that

$$\limsup_{\|\mathbf{x}\|\to\infty} \frac{\ln \|\mathbf{1}_{C_1(\mathbf{x})}\mathbf{f}\|}{\|\mathbf{x}\|} \leq -\rho\rho' m.$$

This completes the proof of Lemma 9.1. □

In the following statement, we treat individual realizations of the random Hamiltonian $\mathbf{H}(\omega)$. This possible owing to our assumption of boundedness of the random amplitude of "impurities", $V(x;\omega), x \in \mathbb{Z}^d$. In a more general case, a similar statement can be proved with probability one with respect to the ensemble of potentials $V(x;\omega)$. In fact, Lemma Lem332 follows from a much more general statement from [St01], so we omit here its proof.

Lemma 9.2. *[Cf. Lemma 3.3.2 in [St01], Section 3.3] Assume that* $\mathbf{H}(\omega)$ *satisfies hypotheses (D) and (E1)–(E4). Then the following properties hold true:*

(A) For spectrally almost every $E \in \mathbf{\Sigma}(\mathbf{H}(\omega))$ *there exists a polynomially bounded eigenfunction corresponding to* E*.*

(B) For every bounded set $I_0 \subset \mathbb{R}$ *there exists a constant* $C = C(M, I_0)$ *such that for every generalized eigenfunction* $\mathbf{\Psi}$ *of* $\mathbf{H}(\omega)$ *corresponding to* $E \in I_0$ *satisfies*

$$\|\mathbf{1}_{\mathbf{C}(\mathbf{u})}\,\mathbf{\Psi}\| \le C\,\|\mathbf{1}_{\mathbf{C}(\mathbf{w})}\,(\mathbf{H}^{\mathbf{\Lambda}}(\omega) - E)^{-1}\mathbf{1}_{\mathbf{C}(\mathbf{u})}\|\;\|\mathbf{1}_{\mathbf{C}(\mathbf{w})}\,\mathbf{\Psi}\|$$

where $\mathbf{H}^{\Lambda}$ *is the restriction of* $\mathbf{H}(\omega)$ *to* $\mathcal{L}_2(\mathbf{\Lambda}(\mathbf{u})$ *with Dirichlet boundary conditions.*

Now we are prepared to prove Theorem 4.1. Indeed, by Lemma 9.2, there is a set $\mathcal{E}_0 \subset I = [0, E_0^*]$ with the following properties:

- $\forall\, E \in \mathcal{E}_0$ there is a polynomially bounded eigenfunction $\mathbf{\Psi}$ of $\mathbf{H}(\omega)$ corresponding to E;
- $I \setminus \mathcal{E}_0$ is a set of measure zero for the spectral resolution of operator $\mathbf{H}(\omega)$.

Further, by Lemma 9.1, every polynomially bounded generalized eigenfunction $\mathbf{\Psi}$ corresponding to $E \in I$ is exponentially decaying, in the $\mathcal{L}_2$-sense, and in particular, $\mathbf{\Psi} \in \mathcal{L}_2(\mathbb{R}^{d\cdot N})$. This means that E is actually an eigenvalue. Moreover, since the Hilbert space $\mathcal{L}_2(\mathbb{R}^{d\cdot N})$ is separable, this implies that the spectrum of $\mathbf{H}(\omega)$ is pure point and, as was just mentioned, all corresponding eigenfunctions decay exponentially in the $\mathcal{L}_2$-sense, as stated in the Theorem 4.1. This concludes the proof. □

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