Random walks and uniform spanning trees: Example
Sheet 2

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The following questions are all based on material from Section 2 of the notes. More difficult questions are marked with a ♠.

Exercise 1. Let \( T \) and \( T' \) be two infinite, bounded degree trees all of whose degrees are at least three. Prove that \( T \) and \( T' \) are rough-isometric.

Exercise 2. Let \( T \) be a \( k \)-regular tree (i.e., the unique tree in which every vertex has degree \( k \)). Prove that
\[
\Phi_E(T) = \frac{k - 2}{k} \quad \text{and} \quad \rho(T) = \frac{2\sqrt{k - 1}}{k} = \sqrt{1 - \Phi_E(T)^2}.
\]

Exercise 3. Let \( G \) be an infinite planar graph all of whose vertex degrees are at least 7. Use Euler’s formula to prove that \( G \) is nonamenable.

Exercise 4. Let \( G \) and \( G' \) be connected, bounded-degree, rough-isometric graphs with isoperimetric profiles \( \Phi_E \) and \( \Phi'_E \) respectively. Prove that there exist positive constants \( c \) and \( C \) such that
\[
c\Phi_E( Ct ) \leq \Phi'_E( t ) \leq C\Phi_E( ct )
\]
for every \( t > 0 \). Prove directly that \( \rho(G) < 1 \) if and only if \( \rho(G') < 1 \) by using Lemma 3.10.

Exercise 5. Fill in the details in the proof of Theorem 3.21 by completing Exercises 27 and 28 from the notes. (Hint: For Exercise 27, reduce to the finite case and use the spectral theorem.)

Exercise 6. Without appealing to Gromov’s theorem, prove that a Cayley graph is recurrent if and only if it satisfies
\[
\sum_{r \geq 1} \frac{r}{Gr(r)} = \infty.
\]

Exercise 7. Construct a connected, locally finite, nonamenable graph for which the walk does not have positive speed. (Such a graph must have unbounded degrees.)
Exercise 8. Construct an infinite, connected, bounded degree graph $G$ with a vertex $v$ such that the random walk $(X_n)_{n \geq 0}$ started from $v$ satisfies $d(v, X_n) \leq C \log n$ with high probability for some constant $C$ as $n \to \infty$. Show that no slower rate of growth of the typical displacement is possible in a bounded degree graph.

Exercise 9. Apply the Varopoulos-Carne inequality to prove Corollaries 3.43-3.45 in the notes.

Exercise 10. Construct a connected, locally finite graph $G$ such that the invariant $\sigma$-algebra $\mathcal{I}$ is trivial but the tail $\sigma$-algebra $\mathcal{T}$ is not.

Exercise 11 (The lamplighter graph.). Let $G$ be a connected, locally finite, simple graph. Let Lamps($V$) be the set of finitely supported functions $\psi : V \to \{0, 1\}$. We define the lamplighter graph LampLighter($G$) to be the graph with vertex set $V \times \text{Lamps}(V)$, and where two vertices $(u, \phi)$ and $(v, \psi)$ are adjacent if and only if $u \sim v$ or $u = v$ and $\phi$ and $\psi$ differ only at $u$. We interpret $(u, \phi)$ as describing the configuration of a collection of lamps, one at each vertex, together with the location of a lamplighter, who is at $u$. At each time step, the lamplighter may either move to a location adjacent to their current location or change the status of the lamp at its current location.

1. Prove that if $G$ is transitive, then LampLighter($G$) is transitive.

2. Prove that if $G$ is infinite then LampLighter($G$) has exponential growth.

3. Prove that if $G$ is transient then LampLighter($G$) has non-trivial invariant $\sigma$-algebra.

4. Prove that LampLighter($\mathbb{Z}^d$) is Liouville if and only if $d \leq 2$.

♦ Exercise 12. Construct an example to prove that the Liouville property is not preserved by rough isometry between connected, bounded degree graphs.

♣♣ Exercise 13. Construct a connected, bounded degree, nonamenable graph that has the Liouville property.

Exercise 14. We say that a function $\phi : \mathbb{Z}^d \to \mathbb{R}$ has sublinear growth if

$$\limsup_{x \to \infty} \frac{\mid \phi(x) \mid}{\mid x \mid} = 0.$$ 

Prove that if $x$ and $y$ are two vertices of $\mathbb{Z}^d$, then there exists a random variable $(Z_n)_{n \geq 0} = ((X_n, Y_n))_{n \geq 0}$ and a random time $T$ such that $(X_n)_{n \geq 0}$ and $(Y_n)_{n \geq 0}$ are both lazy simple random walks on $\mathbb{Z}^d$, $X_n = Y_n$ for every $n \geq T$, and there exists a constant $C_{xy}$ such that $\mathbb{P}(T \geq n) \leq C_{xy}n^{-1}$ for every $n \geq 1$. Deduce that if $\phi : \mathbb{Z}^d \to \mathbb{R}$ is harmonic and has sublinear growth then it is constant.