Reversible Hyperbolic Triangulations

Circle Packing and Random Walk

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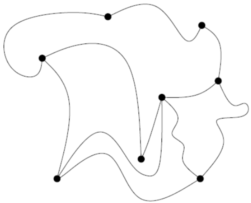

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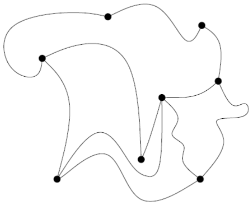

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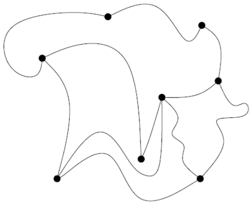

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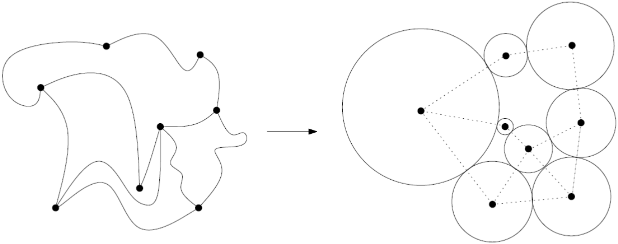

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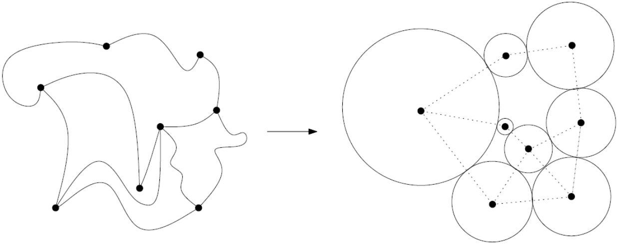

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A finite simple planar graph is a tangency graph of a circle packing.

If G is a triangulation, then the drawing is unique up to Möbius transformations and reflections.

Circle packing definitions

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- Similarly we have packings in the plane $\mathbb{C}$.

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In the bounded degree case, the type of the packing encapsulates probabilistic information: recurrence/transience of the random walk, existence of non-trivial bounded harmonic functions, etc.

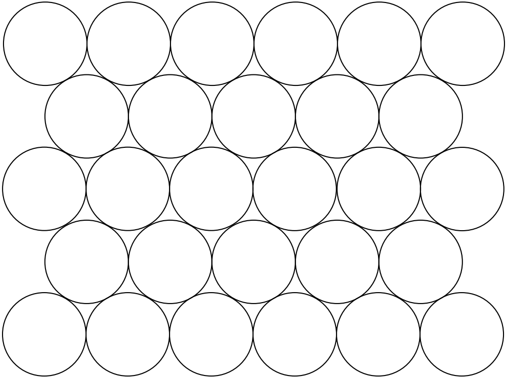
The uniqueness part of the Circle Packing Theorem also extends to the infinite setting.

Theorem (Schramm '91)

Let G be an infinite one-ended simple triangulation circle packed in either $\mathbb{C}$ or $\mathbb{D}$. Such a packing is unique up to Möbius transformations of $\mathbb{C}$ or $\mathbb{D}$ respectively.

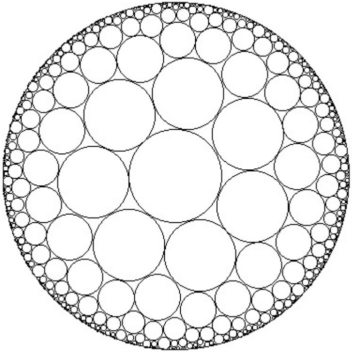
Example 1

The triangular lattice.



Example 2

The 7-regular hyperbolic tessellation circle packed in the unit disc.



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The boundary of the circle packing is a realisation of the Poisson boundary of the random walk: if we know where in the boundary the walk converges to, we know the outcome of every tail event.

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Remark: similar theory developed for **square tilings** by Benjamini and Schramm ('96) and Georgakopoulos ('12), again for bounded degree.

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And, in the hyperbolic case,

- Question 3: Does the walker converge to a point in the boundary of the disc? Is the law of the limit dense and non-atomic almost surely?
- Question 4: Is the unit circle a realisation of the Poisson boundary?

Reversibility

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Finite graphs with roots selected according to the stationary distribution are reversible.

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Theorem (Aldous-Lyons '07)

A random rooted graph (G, ρ) is reversible if and only if its $\deg(\rho)^{-1}$-biasing is unimodular.

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Theorem (Angel, H., Nachmias, Ray '14)

Let (G, ρ) be an ergodic, simple, one-ended, unimodular random rooted triangulation. Then $\mathbb{E}[\deg(\rho)] \geq 6$ and the following are equivalent.

1. $\mathbb{E}[\deg(\rho)] = 6$.
2. (G, ρ) *is CP parabolic almost surely.*
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In particular, (simplifications of) Curien's Stochastic Hyperbolic Triangulations are CP hyperbolic.

Why 6?

In the CP parabolic case:

- Circle packing lets us draw our graph with straight lines in a canonical way.
- Angles of a Euclidean triangle add up to π
- $\longrightarrow$ 'average' angle at a corner is $\pi/3$.
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In the CP hyperbolic case:

- Circle packing lets us draw our graph with hyperbolic geodesics in a canonical way.
- Angles of a hyperbolic triangle add up to less than π
- $\longrightarrow$ 'average' angle at a corner is less than $\pi/3$.
- $\longrightarrow$ expected degree is greater than 6.

- Question 3: Does the walker converge to a point in the boundary of the disc? Is the law of the limit dense and non-atomic almost surely?
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Theorem (Angel, H., Nachmias, Ray '14)

Let (G, ρ) be a unimodular CP hyperbolic random rooted triangulation satisfying $\mathbb{E}[\deg(\rho)^2] < \infty$ and let C be circle packing of G in the disc. Then

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- *The boundary of the disc is a realisation of the Poisson boundary of G.*

Exponential decay of radii

As a consequence of invariant non-amenability, we get the following key intermediate result.

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Easy Corollary 2: the walk has positive speed in the hyperbolic metric.

No atoms in the exit measure

Step 1: There are either no atoms, or one atom with mass one. Suppose there are some atoms.

For each atom a of the exit measure on the disc, let $A_a(v)$ be the probability that a walk started at v exits through a. Let M_v be the maximum of the $A_a(v)$.

If the random walk converges to an atom a, $A_a(X_n)$ converges to 1 almost surely by Lévy's 0-1 law and hence M_{X_n} tends to 1.

By stationarity and ergodicity, M_ρ must have been 1 almost surely to begin with.

Step 2: Now suppose there is a single atom in the exit measure with weight one.

Applying a Möbius transformation sending this atom to infinity, we get a circle packing of G in the upper half plane such that the random walk converges to infinity almost surely: such a circle packing is unique up to translation and dilation.

So now we have a way of canonically drawing the triangulation with straight lines! The same ideas as earlier imply that the expected degree must be 6, contradicting our characterisation of CP hyperbolicity.

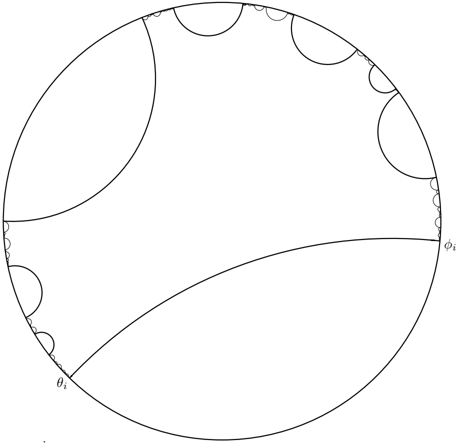
Full support

Suppose the exit measure does not have full support.

We will define a mass transport on G in which each vertex sends a mass of at most one, but some vertices receive infinite mass, contradicting the mass transport principle.

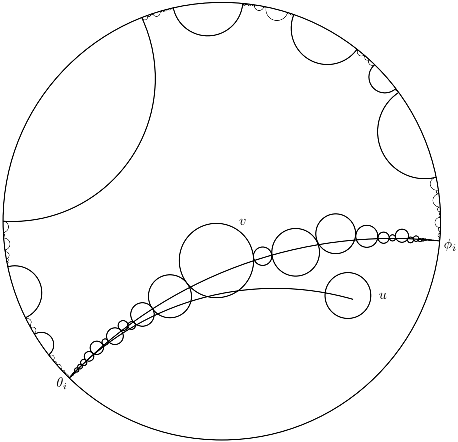
The transport will be defined in terms of the hyperbolic geometry and the support of the exit measure, so that it will not depend on the choice of the packing (and so will really be a mass transport).

The complement of the support of the exit measure may be written as a union of disjoint open intervals (θ_i, ϕ_i) in the circle. Let's draw the hyperbolic geodesic γ_i between the endpoints of each such interval.



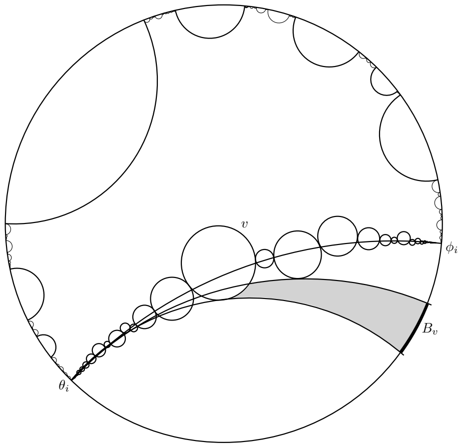
Write A_i for the set of circles enclosed by the geodesic between θ_i and ϕ_i.

Transport mass one from each u in A_i to the first circle intersected by the geodesic from the hyperbolic centre of u to θ_i that also intersects γ_i.

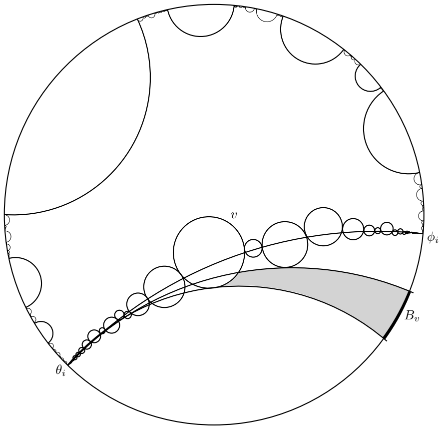


It might be that no such circle exists, in which case u sends no mass.

Consider the set of angles $B_v \subset (\theta_i, \phi_i)$ such that v is the first circle intersected by the geodesic from θ to θ_i that also intersects γ_i. For each v, this set is an interval.



The union of the B_v's over all v intersecting γ_i is an interval of positive length, and hence, since there are only countably many circles, one of the intervals B_v has positive length.



Such a vertex receives infinite mass, since it is sent mass by every vertex with centres in some open neighbourhood of the boundary interval.

Thank you!