STATISTICS Part IB
Correction Lent 2020

A note on the existence of LR tests of exact size α

There is an error in the notes by Prof. Spiegelhalter, which was repeated in lecture. If $X_1, \dots, X_n$ are i.i.d. with probability density function f, and $H_0 : f = f_0$, $H_1 : f = f_1$ are two simple hypotheses, the continuity of f_0 and f_1 does **not** imply that there exists a likelihood ratio test of exact size α, for any $\alpha \in (0, 1)$.

However, one can check that there always exists a *randomised* likelihood ratio test with exact size α. If f_0 and f_1 are nonzero on the same sets, then the ratio $\Lambda_x(H_0; H_1) = f_1(x)/f_0(x)$ is a well defined random variable with distribution function F under H_0. Then, given some $\alpha \in (0, 1)$, we can define $k = \inf\{u \in \mathbb{R}; 1 - F(u) \leq \alpha \}$, Define a test which rejects H_0 if $\Lambda_x(H_0; H_1) > k$, fails to reject H_0 if $\Lambda_x(H_0; H_1) < k$, and if $\Lambda_x(H_0; H_1) = k$, it has a fixed probability C of rejecting H_0. Then, we can always choose a C such that the test has Type I error probability exactly α.