Time Series — Examples Sheet

This is the examples sheet for the M. Phil. course in Time Series. A copy can be found at: \url{http://www.statslab.cam.ac.uk/~rrw1/timeseries/}

Throughout, unless otherwise stated, the sequence \( \{\epsilon_t\} \) is white noise, variance \( \sigma^2 \).
1. Find the Yule-Walker equations for the AR(2) process

\[ X_t = \frac{1}{3}X_{t-1} + \frac{2}{5}X_{t-2} + \epsilon_t. \]

Hence show that it has autocorrelation function

\[ \rho_k = \frac{16}{21} \left( \frac{2}{3} \right)^{|k|} + \frac{5}{21} \left( -\frac{1}{3} \right)^{|k|}, \quad k \in \mathbb{Z}. \]

[ The Yule-Walker equations are

\[ \rho_k - \frac{1}{3}\rho_{k-1} - \frac{2}{5}\rho_{k-2} = 0, \quad k \geq 2. \]

On trying \( \rho_k = A\lambda^k \), we require \( \lambda^2 - \frac{1}{3}\lambda - \frac{2}{5} = 0 \). This has roots \( \frac{2}{3} \) and \( -\frac{1}{3} \), so

\[ \rho_k = A \left( \frac{2}{3} \right)^{|k|} + B \left( -\frac{1}{3} \right)^{|k|}, \]

where \( \rho_0 = A + B = 1 \). We also require \( \rho_1 = \frac{1}{3} + \frac{2}{5}\rho_1 \). Hence \( \rho_1 = \frac{3}{7} \), and thus we require \( \frac{2}{3}A - \frac{1}{3}B = \frac{3}{7} \). These give \( A = \frac{16}{21}, \ B = \frac{5}{21}. \) ]
2. Let \( X_t = A \cos(\Omega t + U) \), where \( A \) is an arbitrary constant, \( \Omega \) and \( U \) are independent random variables, \( \Omega \) has distribution function \( F \) over \([0, \pi]\), and \( U \) is uniform over \([0, 2\pi]\). Find the autocorrelation function and spectral density function of \( \{X_t\} \). Hence show that, for any positive definite set of covariances \( \{\gamma_k\} \), there exists a process with autocovariances \( \{\gamma_k\} \) such that every realization is a sine wave.

[Use the following definition: \( \{\gamma_k\} \) are positive definite if there exists a nondecreasing function \( F \) such that \( \gamma_k = \int_{-\pi}^{\pi} e^{ik\omega} d\bar{F}(\omega) \).]

\[
\mathbb{E}[X_{t+\tau} X_t] = \frac{1}{2\pi} \int_0^{2\pi} A \cos(\Omega(t+u) + \omega) du = \frac{1}{2\pi} A \sin(\Omega(t+u)) \bigg|_0^{2\pi} = 0
\]

\[
\mathbb{E}[X_{t+\tau} X_t] = \frac{1}{2\pi} \int_0^{\pi} \int_0^{2\pi} A \cos(\Omega(t+s) + u) A \cos(\Omega(t+u)) du dF(\Omega)
\]

\[
= \frac{1}{4\pi} \int_0^{\pi} \int_0^{2\pi} A^2 \cos(\Omega s) du dF(\Omega)
\]

\[
= \frac{1}{4} \left[ e^{i\Omega s} + e^{-i\Omega s} \right] dF(\Omega)
\]

\[
= \frac{1}{4} A^2 \int_{-\pi}^{\pi} e^{i\Omega s} d\bar{F}(\Omega)
\]

where we define over the range \([-\pi, \pi]\) the nondecreasing function \( \bar{F} \), by \( \bar{F}(-\Omega) = F(\pi) - F(\Omega) \) and \( \bar{F}(\Omega) = F(\Omega) + F(\pi) - 2F(0) \), \( \Omega \in [0, \pi] \).]
3. Find the spectral density function of the AR(2) process

\[ X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \epsilon_t . \]

What conditions on \((\phi_1, \phi_2)\) are required for this process to be an indeterministic second order stationary? Sketch in the \((\phi_1, \phi_2)\) plane the stationary region.

\[
\begin{align*}
\text{We have} \\
&f_X(\omega)\left|1 - \phi_1 e^{i\omega} - \phi_2 e^{2i\omega}\right|^2 = \sigma^2 / \pi \\
\text{Hence} \\
f_X(\omega) = \frac{\sigma^2}{\pi [1 + \phi_1^2 + \phi_2^2 + 2(-\phi_1 + \phi_1 \phi_2) \cos(\omega) - 2\phi_2 \cos(2\omega)]}
\end{align*}
\]

The Yule-Walker equations have solution of the form \(\rho_k = A\lambda_1^k + B\lambda_2^k\) where \(\lambda_1, \lambda_2\) are roots of

\[ g(\lambda) = \lambda^2 - \phi_1 \lambda - \phi_2 = 0. \]

The roots are \(\lambda = \left[ \phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2} \right] / 2\). To be indeterministic second order stationary these roots must have modulus less than 1. If \(\phi_1^2 + 4\phi_2 > 0\) then the roots are real and lie in the range \([-1, 1]\) if and only if \(g(-1) > 0\) and \(g(1) > 0\), i.e., \(\phi_1 + \phi_2 < 1, \phi_1 - \phi_2 > -1\). If \(\phi_1^2 + 4\phi_2 < 0\) then the roots are complex and their product must be less than 1, i.e., \(\phi_2 > -1\). The union of these two regions, corresponding to possible \((\phi_1, \phi_2)\) for real and imaginary roots, is simply the triangular region

\[ \phi_1 + \phi_2 < 1, \quad \phi_1 - \phi_2 > -1, \quad \phi_2 \geq -1. \]
4. For a stationary process define the covariance generating function

\[ g(z) = \sum_{k=-\infty}^{\infty} \gamma_k z^k, \quad |z| < 1. \]

Suppose \( \{X_t\} \) satisfies \( X = C(B) \varepsilon \), that is, it has the Wold representation

\[ X_t = \sum_{r=0}^{\infty} c_r \varepsilon_{t-r}, \]

where \( \{c_r\} \) are constants satisfying \( \sum_{r=0}^{\infty} c_r^2 < \infty \) and \( C(z) = \sum_{r=0}^{\infty} c_r z^r \). Show that

\[ g(z) = C(z)C(z^{-1})\sigma^2. \]

Explain how this can be used to derive autocovariances for the ARMA\((p, q)\) model. Hence show that for ARMA\((1, 1)\), \( \rho_2^2 = \rho_1 \rho_3 \). How might this fact be useful?

[ We have

\[ \gamma_k = \mathbb{E}X_t X_{t+k} = \mathbb{E} \left[ \sum_{r=0}^{\infty} c_r \varepsilon_{t-r} \sum_{s=0}^{\infty} c_s \varepsilon_{t+k-s} \right] \]

\[ = \sigma^2 \sum_{r=0}^{\infty} c_r c_{k+r} \]

Now

\[ C(z)C(z^{-1}) = \sum_{r=0}^{\infty} c_r z^r \sum_{s=0}^{\infty} c_s z^{-s} \]

The coefficients of \( z^k \) and \( z^{-k} \) are clearly

\[ c_k c_0 + c_{k+1} c_1 + c_{k+2} c_3 + \cdots \]

from which the result follows.

For the ARMA\((p, q)\) model \( \phi(B)X = \theta(B) \varepsilon \) or

\[ X = \frac{\theta(B)}{\phi(B)} \varepsilon \]

where \( \phi \) and \( \theta \) are polynomials of degrees \( p \) and \( q \) in \( z \). Hence

\[ C(z) = \frac{\theta(z)}{\phi(z)} \]
and $\gamma_k$ can be found as the coefficient of $z^k$ in the power series expansion of $\sigma^2 \theta(z)\theta(1/z)/\phi(z)\phi(1/z)$. For ARMA(1, 1) this is

$$\sigma^2(1 + \theta z)(1 + \theta z^{-1})(1 + \phi z + \phi^2 z^2 + \cdots)(1 + \phi z^{-1} + \phi^2 z^{-2} + \cdots)$$

from which we have

$$\gamma_1 = \left( \theta(1 + \phi^2 + \phi^4 + \cdots) + (\phi + \phi^3 + \phi^5 + \cdots)(1 + \theta^2) + \theta(\phi^2 + \phi^4 + \phi^6 + \cdots) \right) \sigma^2$$

$$= \frac{\theta + \phi(1 + \theta^2) + \phi^2 \theta}{1 - \phi^2} \sigma^2$$

and similarly

$$\gamma_2 = \phi^2 \theta + \phi(1 + \theta^2) + \phi^2 \theta \sigma^2$$

and

$$\gamma_3 = \frac{\phi^2 \theta + \phi(1 + \theta^2) + \phi^2 \theta}{1 - \phi^2} \sigma^2$$

Hence $\rho_2^2 = \rho_1 \rho_3$. This might be used as a diagnostic to test the appropriateness of an ARMA(1, 1) model, by reference to the correlogram, where we would expect to see $r_2^2 = r_1 r_3$. ]
5. Consider the ARMA(2, 1) process defined as

\[ X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \epsilon_t + \theta_1 \epsilon_{t-1}. \]

Show that the coefficients of the Wold representation satisfy the difference equation

\[ c_k = \phi_1 c_{k-1} + \phi_2 c_{k-2}, \quad k \geq 2, \]

and hence that

\[ c_k = Az_1^{-k} + Bz_2^{-k}, \]

where \( z_1 \) and \( z_2 \) are zeros of \( \phi(z) = 1 - \phi_1 z - \phi_2 z^2 \), and \( A \) and \( B \) are constants. Explain how in principle one could find \( A \) and \( B \).

[ The recurrence is produced by substituting \( X_t = \sum_{r=0}^{\infty} c_r \epsilon_{t-r} \) into the defining equation, and similarly for \( X_{t-1} \) and \( X_{t-2} \), multiplying by \( \epsilon_{t-k} \), \( k \geq 2 \), and taking expected value.

The general solution to such a second order linear recurrence relation is of the form given and we find \( A \) and \( B \) by noting that

\[ X_t = \phi_1 (\phi_1 X_{t-2} + \phi_2 X_{t-3} + \epsilon_{t-1} + \theta_1 \epsilon_{t-2}) + \phi_2 X_{t-2} + \epsilon_t + \theta_1 \epsilon_{t-1} \]

so that \( c_0 = 1 \) and \( c_1 = \theta_1 + \phi_1 \). Hence \( A + B = 1 \) and \( Az_1^{-1} + Bz_2^{-1} = \theta_1 + \phi_1 \). These can be solved for \( A \) and \( B \). ]
6. Suppose 
\[ Y_t = X_t + \epsilon_t, \quad X_t = \alpha X_{t-1} + \eta_t, \]
where \(\{\epsilon_t\}\) and \(\{\eta_t\}\) are independent white noise sequences with common variance \(\sigma^2\). Show that the spectral density function of \(\{Y_t\}\) is
\[
f_Y(\omega) = \frac{\sigma^2}{\pi} \left\{ \frac{2 - 2\alpha \cos \omega + \alpha^2}{1 - 2\alpha \cos \omega + \alpha^2} \right\}.
\]
For what values of \(p, d, q\) is the autocovariance function of \(\{Y_t\}\) identical to that of an ARIMA\((p, d, q)\) process?

\[
\begin{align*}
    f_Y(\omega) &= f_X(\omega) + f_\epsilon(\omega) = \frac{1}{|1 - \alpha e^{i\omega}|^2} f_\eta(\omega) + f_\epsilon(\omega) \\
    &= \frac{\sigma^2}{\pi} \left\{ \frac{1}{1 - 2\alpha \cos \omega + \alpha^2} + 1 \right\} = \frac{\sigma^2}{\pi} \left\{ \frac{2 - 2\alpha \cos \omega + \alpha^2}{1 - 2\alpha \cos \omega + \alpha^2} \right\}.
\end{align*}
\]
We recognise this as the spectral density of an ARMA\((1, 1)\) model. E.g., \(Z_t - \alpha Z_{t-1} = \xi_t - \theta \xi_{t-1}\), choosing \(\theta\) and \(\sigma^2_\xi\) such that
\[
(1 - 2\theta \cos \omega + \theta^2)\sigma^2_\xi = (\sigma^2/\pi)(2 - 2\alpha \cos \omega + \alpha^2)
\]
I.e., choosing \(\theta\) such that \((1 + \theta^2)/\theta = (2 + \alpha^2)/\alpha\). \]
7. Suppose $X_1, \ldots, X_T$ are values of a time series. Prove that

$$\left\{ \hat{\gamma}_0 + 2 \sum_{k=1}^{T-1} \hat{\gamma}_k \right\} = 0,$$

where $\hat{\gamma}_k$ is the usual estimator of the $k$th order autocovariance,

$$\hat{\gamma}_k = \frac{1}{T} \sum_{t=k+1}^{T} (X_t - \bar{X})(X_{t-k} - \bar{X}).$$

Hint: Consider $0 = \sum_{t=1}^{T} (X_t - \bar{X})$.

Hence deduce that not all ordinates of the correlogram can have the same sign.

Suppose $f(\cdot)$ is the spectral density and $I(\cdot)$ the periodogram. Suppose $f$ is continuous and $f(0) \neq 0$. Does $\mathbb{E}I(2\pi/T) \to f(0)$ as $T \to \infty$?

[ The results follow directly from

$$\frac{1}{T} \left[ \sum_{t=1}^{T} (X_t - \bar{X}) \right]^2 = 0.$$ 

Note that formally,

$$I(0) = \hat{\gamma}_0 + 2 \sum_{k=1}^{T-1} \hat{\gamma}_k = 0.$$ 

so it might appear that $\mathbb{E}I(2\pi/T) \to I(0) \neq f(0)$ as $T \to \infty$. However, this would be mistaken. It is a theorem that as $T \to \infty$, $I(\omega_j) \sim f(\omega_j) \chi^2/2$. So for large $T$, $\mathbb{E}I(2\pi/T) \approx f(0)$. ]
8. Suppose $I(\cdot)$ is the periodogram of $\epsilon_1, \ldots, \epsilon_T$, where these are i.i.d. $N(0,1)$ and $T = 2m + 1$. Let $\omega_j, \omega_k$ be two distinct Fourier frequencies, Show that $I(\omega_j)$ and $I(\omega_k)$ are independent random variables. What are their distributions?

If it is suspected that $\{\epsilon_t\}$ departs from white noise because of the presence of a single harmonic component at some unknown frequency $\omega$ a natural test statistic is the maximum periodogram ordinate

$$T = \max_{j=1,\ldots,m} I(\omega_j).$$

Show that under the hypothesis that $\{\epsilon_t\}$ is white noise

$$P(T > t) = 1 - \left\{1 - \exp\left(-\frac{\pi t}{\sigma^2}\right)\right\}^m.$$

[ The independence of $I(\omega_j)$ and $I(\omega_k)$ was proved in lectures. Their distributions are $(\sigma^2/2\pi)\chi^2_2$, which is equivalent to the exponential distribution with mean $\sigma^2/\pi$. Hence the probability that the maximum is less than $t$ is the probability that all are, i.e.,

$$P(T < t) = \left\{1 - \exp\left(-\frac{\pi t}{\sigma^2}\right)\right\}^m.$$
]
9. Complete this sketch of the fast Fourier transform. From data \(X_0, \ldots, X_T\), with \(T = 2^M - 1\), we want to compute the \(2^M - 1\) ordinates of the periodogram

\[
I(\omega_j) = \frac{1}{\pi T} \left| \sum_{t=0}^{T} X_t e^{it2\pi j/2^M} \right|^2, \quad j = 1, \ldots, 2^M - 1.
\]

This requires order \(T\) multiplications for each \(j\) and so order \(T^2\) multiplications in all. However,

\[
\sum_{t=0,1,\ldots,2^M-1} X_t e^{it2\pi j/2^M} = \sum_{t=0,2,\ldots,2^M-2} X_t e^{it2\pi j/2^M} + \sum_{t=1,3,\ldots,2^M-1} X_t e^{it2\pi j/2^M}
\]

\[
= \sum_{t=0,1,\ldots,2^M-1-1} X_{2t} e^{i(2t)2\pi j/2^M} + \sum_{t=0,1,\ldots,2^M-1-1} X_{2t+1} e^{i(2t+1)2\pi j/2^M}
\]

Note that the value of either sum on the right hand side at \(j = k\) is the complex conjugate of its value at \(j = (2^M - 1 - k)\); so these sums need only be computed for \(j = 1, \ldots, 2^M - 2\). Thus we have two sums, each of which is similar to the sum on the left hand side, but for a problem half as large. Suppose the computational effort required to work out each right hand side sum (for all \(2^M - 2\) values of \(j\)) is \(\Theta(M - 1)\). The sum on the left hand side is obtained (for all \(2^M - 1\) values of \(j\)) by combining the right hand sums, with further computational effort of order \(2^M - 1\). Explain

\[
\Theta(M) = a2^{M-1} + 2\Theta(M - 1).
\]

Hence deduce that \(I(\cdot)\) can be computed (by the FFT) in time \(T \log_2 T\).

[ The derivation of the recurrence for \(\Theta(M)\) should be obvious. We have \(\Theta(1) = 1\), and hence \(\Theta(M) = aM2^M = O(T \log_2 T)\). ]
10. Suppose we have the ARMA(1, 1) process
\[ X_t = \phi X_{t-1} + \epsilon_t + \theta \epsilon_{t-1}, \]
with \(|\phi| < 1, |\theta| < 1, \phi + \theta \neq 0\), observed up to time \(T\), and we want to calculate \(k\)-step ahead forecasts \(\hat{X}_{T,k}\), \(k \geq 1\).

Derive a recursive formula to calculate \(\hat{X}_{T,k}\) for \(k = 1\) and \(k = 2\).

\[
\begin{align*}
\hat{X}_{T,1} &= \phi X_T + \hat{\epsilon}_{T+1} + \theta \hat{\epsilon}_T = \phi X_T + \theta (X_T - \hat{X}_{T-1,1}) \\
\hat{X}_{T,2} &= \phi \hat{X}_{T,1} + \hat{\epsilon}_{T+2} + \theta \hat{\epsilon}_{T+1} = \phi \hat{X}_{T,1}
\end{align*}
\]
Consider the stationary scalar-valued process \( \{X_t\} \) generated by the moving average, \( X_t = \epsilon_t - \theta \epsilon_{t-1} \).

Determine the linear least-square predictor of \( X_t \), in terms of \( X_{t-1}, X_{t-2}, \ldots \).

We can directly apply our results to give

\[
\hat{X}_{t-1,1} = -\theta \hat{\epsilon}_{t-1} \\
= -\theta [X_{t-1} - \hat{X}_{t-2,1}] \\
= -\theta X_{t-1} + \theta \hat{X}_{t-2,1} \\
= -\theta X_{t-1} + \theta [-\theta X_{t-2} + \theta \hat{X}_{t-3,1}] \\
= -\theta X_{t-1} - \theta^2 X_{t-2} - \theta^3 X_{t-3} - \cdots
\]

Alternatively, take the linear predictor as \( \hat{X}_{t-1,1} = \sum_{r=1}^{\infty} a_r X_{t-r} \) and seek to minimize \( \mathbb{E}[X_t - \hat{X}_{t-1,1}]^2 \). We have

\[
\mathbb{E}[X_t - \hat{X}_{t-1,1}]^2 = \mathbb{E} \left[ \epsilon_t - \theta \epsilon_{t-1} - \sum_{r=1}^{\infty} a_r (\epsilon_{t-r} - \theta \epsilon_{t-r-1}) \right]^2 \\
= \sigma^2 \left[ 1 + (\theta + a_1)^2 + (\theta a_1 - a_2)^2 + (\theta a_2 - a_3)^2 + \cdots \right]
\]

Note that all terms, but the first, are minimized to 0 by taking \( a_r = -\theta^r \).
12. Consider the ARIMA(0, 2, 2) model

\[(I - B)^2X = (I - 0.81B + 0.38B^2)\epsilon\]

where \(\{\epsilon_t\}\) is white noise with variance 1.

(a) With data up to time \(T\), calculate the \(k\)-step ahead optimal forecast of \(\hat{X}_{T,k}\) for all \(k \geq 1\). By giving a general formula relating \(\hat{X}_{T,k}\), \(k \geq 3\), to \(\hat{X}_{T,1}\) and \(\hat{X}_{T,2}\), determine the curve on which all these forecasts lie.

The model is

\[X_t = 2X_{t-1} - X_{t-2} + \epsilon_t - 0.81\epsilon_{t-1} + 0.38\epsilon_{t-2}\]

Hence

\[\hat{X}_{T,1} = 2X_T - X_{T-1} + \hat{\epsilon}_{T+1} - 0.81\hat{\epsilon}_T + 0.38\hat{\epsilon}_{T-1}\]
\[= 2X_T - X_{T-1} - 0.81[X_T - \hat{X}_{T-1,1}] + 0.38[X_T - \hat{X}_{T-2,1}]\]

and similarly

\[\hat{X}_{T,2} = 2\hat{X}_{T,1} - X_T + 0.38[X_T - \hat{X}_{T-1,1}]\]
\[\hat{X}_{T,k} = 2\hat{X}_{T,k-1} - \hat{X}_{T,k-2}, \; k \geq 3\).

This implies that the forecasts lie on a straight line.

(b) Suppose now that \(T = 95\). Calculate numerically the forecasts \(\hat{X}_{95,k}\), \(k = 1, 2, 3\) and their mean squared prediction errors when the last five observations are \(X_{91} = 15.1, X_{92} = 15.8, X_{93} = 15.9, X_{94} = 15.2, X_{95} = 15.9\).

[You will need estimates for \(\epsilon_{94}\) and \(\epsilon_{95}\). Start by assuming \(\epsilon_{91} = \epsilon_{92} = 0\), then calculate \(\hat{\epsilon}_{93} = \epsilon_{93} = X_{93} - \hat{X}_{92,1}\), and so on, until \(\epsilon_{94}\) and \(\epsilon_{95}\) are obtained.]

[Using the above formulae we obtain]

<table>
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<th>(t)</th>
<th>(X_t)</th>
<th>(\hat{X}_{t,1})</th>
<th>(\hat{X}_{t,2})</th>
<th>(\hat{X}_{t,3})</th>
<th>(\epsilon_t)</th>
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<td>15.636</td>
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</tbody>
</table>

Now

\[X_{T+k} = \sum_{r=0}^{\infty} c_r \epsilon_{T+k-r}\] and \(\hat{X}_{T,k} = \sum_{r=k}^{\infty} c_r \epsilon_{T+k-r}\).
Thus
\[
\mathbb{E} \left[ X_{T+k} - \hat{X}_{T,k} \right]^2 = \sigma^2 \epsilon \sum_{r=0}^{k-1} c_r^2.
\]
where $\sigma^2 = 1$. Now
\[
X_T = \epsilon_T + (2 - 0.81)\epsilon_{T-1} + (-2(0.81) - 1 + 0.38)\epsilon_{T-2} + \cdots
\]
Hence the mean square errors of $\hat{X}_{95,1}$, $\hat{X}_{95,2}$, $\hat{X}_{95,3}$ are respectively 1, 1.416, 5.018.
13. Consider the state space model,

\[ X_t = S_t + v_t, \]
\[ S_t = S_{t-1} + w_t, \]

where \( X_t \) and \( S_t \) are both scalars, \( X_t \) is observed, \( S_t \) is unobserved, and \( \{v_t\}, \{w_t\} \) are Gaussian white noise sequences with variances \( V \) and \( W \) respectively. Write down the Kalman filtering equations for \( \hat{S}_t \) and \( P_t \).

Show that \( P_t \equiv P \) (independently of \( t \)) if and only if \( P^2 + PW = WV \), and show that in this case the Kalman filter for \( \hat{S}_t \) is equivalent to exponential smoothing.

\[ \text{This is the same as Section 8.4 of the notes.} \]

\[ F_t = 1, \ G_t = 1, \ V_t = V, \ W_t = W. \]
\[ R_t = P_{t-1} + W. \]

So if \( (S_{t-1} \mid X_1, \ldots, X_{t-1}) \sim N \left( \hat{S}_{t-1}, P_{t-1} \right) \) then \( (S_t \mid X_1, \ldots, X_t) \sim N \left( \hat{S}_t, P_t \right) \), where

\[ \hat{S}_t = \hat{S}_{t-1} + R_t \frac{R_t^2}{V + R_t} (X_t - \hat{S}_{t-1}) \]
\[ P_t = R_t - \frac{R_t^2}{V + R_t} = \frac{VR_t}{V + R_t} = \frac{V(P_{t-1} + W)}{V + P_{t-1} + W}. \]

\( P_t \) is constant if \( P_t \equiv P \), where \( P \) is the positive root of \( P^2 + WP - WV = 0. \)

In this case \( \hat{S}_t \) behaves like \( \hat{S}_t = (1 - \alpha) \sum_{r=0}^{\infty} \alpha^r X_{t-r} \), where \( \alpha = V/(V + W + P). \)

This is simple exponential smoothing.