Time Series — Examples Sheet

This is the examples sheet for the M. Phil. course in Time Series. A copy can be found at: http://www.statslab.cam.ac.uk/~rrw1/timeseries/

Throughout, unless otherwise stated, the sequence \( \{\epsilon_t\} \) is white noise, variance \( \sigma^2 \).
1. Find the Yule-Walker equations for the AR(2) process

\[ X_t = \frac{1}{3} X_{t-1} + \frac{2}{3} X_{t-2} + \epsilon_t. \]

Hence show that it has autocorrelation function

\[ \rho_k = \frac{16}{21} \left( \frac{2}{3} \right)^{|k|} + \frac{5}{21} \left( -\frac{1}{3} \right)^{|k|}, \quad k \in \mathbb{Z}. \]
2. Let $X_t = A \cos(\Omega t + U)$, where $A$ is an arbitrary constant, $\Omega$ and $U$ are independent random variables, $\Omega$ has distribution function $F$ over $[0, \pi]$, and $U$ is uniform over $[0, 2\pi]$. Find the autocorrelation function and spectral density function of $\{X_t\}$. Hence show that, for any positive definite set of covariances $\{\gamma_k\}$, there exists a process with autocovariances $\{\gamma_k\}$ such that every realization is a sine wave.

[Use the following definition: $\{\gamma_k\}$ are positive definite if there exists a nondecreasing function $F$ such that $\gamma_k = \int_{-\pi}^{\pi} e^{ik\omega}dF(\omega).$]
3. Find the spectral density function of the AR(2) process

\[ X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \epsilon_t. \]

What conditions on \((\phi_1, \phi_2)\) are required for this process to be an indeterministic second order stationary? Sketch in the \((\phi_1, \phi_2)\) plane the stationary region.
4. For a stationary process define the covariance generating function

\[ g(z) = \sum_{k=-\infty}^{\infty} \gamma_k z^k, \quad |z| < 1. \]

Suppose \( \{X_t\} \) satisfies \( X = C(B)\epsilon \), that is, it has the Wold representation

\[ X_t = \sum_{r=0}^{\infty} c_r \epsilon_{t-r}, \]

where \( \{c_r\} \) are constants satisfying \( \sum_0^{\infty} c_r^2 < \infty \) and \( C(z) = \sum_{r=0}^{\infty} c_r z^r \). Show that

\[ g(z) = C(z)C(z^{-1})\sigma^2. \]

Explain how this can be used to derive autocovariances for the ARMA\((p, q)\) model. Hence show that for ARMA\((1, 1)\), \( \rho_2^2 = \rho_1 \rho_3 \). How might this fact be useful?
5. Consider the ARMA(2, 1) process defined as
\[ X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \epsilon_t + \theta_1 \epsilon_{t-1}. \]
Show that the coefficients of the Wold representation satisfy the difference equation
\[ c_k = \phi_1 c_{k-1} + \phi_2 c_{k-2}, \quad k \geq 2, \]
and hence that
\[ c_k = A z_1^{-k} + B z_2^{-k}, \]
where \( z_1 \) and \( z_2 \) are zeros of \( \phi(z) = 1 - \phi_1 z - \phi_2 z^2 \), and \( A \) and \( B \) are constants. Explain how in principle one could find \( A \) and \( B \).
6. Suppose
\[ Y_t = X_t + \epsilon_t, \quad X_t = \alpha X_{t-1} + \eta_t, \]
where \( \{\epsilon_t\} \) and \( \{\eta_t\} \) are independent white noise sequences with common variance \( \sigma^2 \). Show that the spectral density function of \( \{Y_t\} \) is
\[ f_Y(\omega) = \frac{\sigma^2}{\pi} \left\{ \frac{2 - 2\alpha \cos \omega + \alpha^2}{1 - 2\alpha \cos \omega + \alpha^2} \right\}. \]

For what values of \( p, d, q \) is the autocovariance function of \( \{Y_t\} \) identical to that of an ARIMA\((p, d, q)\) process?
7. Suppose $X_1, \ldots, X_T$ are values of a time series. Prove that
\[
\left\{ \hat{\gamma}_0 + 2 \sum_{k=1}^{T-1} \hat{\gamma}_k \right\} = 0,
\]
where $\hat{\gamma}_k$ is the usual estimator of the $k$th order autocovariance,
\[
\hat{\gamma}_k = \frac{1}{T} \sum_{t=k+1}^{T} (X_t - \bar{X})(X_{t-k} - \bar{X}).
\]

Hint: Consider $0 = \sum_{t=1}^{T} (X_t - \bar{X})$.
Hence deduce that not all ordinates of the correlogram can have the same sign.

Suppose $f(\cdot)$ is the spectral density and $I(\cdot)$ the periodogram. Suppose $f$ is continuous and $f(0) \neq 0$. Does $\mathbb{E} I(2\pi/T) \to f(0)$ as $T \to \infty$?
8. Suppose $I(\cdot)$ is the periodogram of $\epsilon_1, \ldots, \epsilon_T$, where these are i.i.d. $N(0, 1)$ and $T = 2m + 1$. Let $\omega_j, \omega_k$ be two distinct Fourier frequencies. Show that $I(\omega_j)$ and $I(\omega_k)$ are independent random variables. What are their distributions?

If it is suspected that $\{\epsilon_t\}$ departs from white noise because of the presence of a single harmonic component at some unknown frequency $\omega$ a natural test statistic is the maximum periodogram ordinate

$$T = \max_{j=1,\ldots,m} I(\omega_j).$$

Show that under the hypothesis that $\{\epsilon_t\}$ is white noise

$$P(T > t) = 1 - \left\{1 - \exp\left(-\pi t/\sigma^2\right)\right\}^m.$$
9. Complete this sketch of the fast Fourier transform. From data \(X_0, \ldots, X_T\), with \(T = 2^M - 1\), we want to compute the \(2^{M-1}\) ordinates of the periodogram

\[
I(\omega_j) = \frac{1}{\pi T} \left| \sum_{t=0}^{T} X_t e^{it2\pi j/2^M} \right|^2, \quad j = 1, \ldots, 2^{M-1}.
\]

This requires order \(T\) multiplications for each \(j\) and so order \(T^2\) multiplications in all. However,

\[
\sum_{t=0,1,\ldots,2^M-1} X_t e^{it2\pi j/2^M} = \sum_{t=0,2,\ldots,2^{M-2}} X_t e^{it2\pi j/2^M} + \sum_{t=1,3,\ldots,2^{M-1}} X_t e^{it2\pi j/2^M}
\]

\[
= \sum_{t=0,1,\ldots,2^{M-1}-1} X_{2t} e^{i2t2\pi j/2^M} + \sum_{t=0,1,\ldots,2^{M-1}-1} X_{2t+1} e^{i(2t+1)2\pi j/2^M}
\]

\[
= \sum_{t=0,1,\ldots,2^{M-1}-1} X_{2t} e^{i2\pi j/2^{M-1}} + e^{i2\pi j/2^{M-1}} \sum_{t=0,1,\ldots,2^{M-1}-1} X_{2t+1} e^{i2\pi j/2^{M-1}}.
\]

Note that the value of either sum on the right hand side at \(j = k\) is the complex conjugate of its value at \(j = (2^{M-1} - k)\); so these sums need only be computed for \(j = 1, \ldots, 2^{M-2}\). Thus we have two sums, each of which is similar to the sum on the left hand side, but for a problem half as large. Suppose the computational effort required to work out each right hand side sum (for all \(2^{M-2}\) values of \(j\)) is \(\Theta(M - 1)\). The sum on the left hand side is obtained (for all \(2^{M-1}\) values of \(j\)) by combining the right hand sums, with further computational effort of order \(2^{M-1}\). Explain

\[
\Theta(M) = a2^{M-1} + 2\Theta(M - 1).
\]

Hence deduce that \(I(\cdot)\) can be computed (by the FFT) in time \(T\log_2 T\).
Suppose we have the ARMA(1, 1) process

\[ X_t = \phi X_{t-1} + \epsilon_t + \theta \epsilon_{t-1}, \]

with \(|\phi| < 1\), \(|\theta| < 1\), \(\phi + \theta \neq 0\), observed up to time \(T\), and we want to calculate \(k\)-step ahead forecasts \(\hat{X}_{T,k}\), \(k \geq 1\).

Derive a recursive formula to calculate \(\hat{X}_{T,k}\) for \(k = 1\) and \(k = 2\).
11. Consider the stationary scalar-valued process \( \{X_t\} \) generated by the moving average, \( X_t = \epsilon_t - \theta \epsilon_{t-1} \).

Determine the linear least-square predictor of \( X_t \), in terms of \( X_{t-1}, X_{t-2}, \ldots \).
12. Consider the ARIMA(0, 2, 2) model

\[(I - B)^2X = (I - 0.81B + 0.38B^2)\epsilon\]

where \(\{\epsilon_t\}\) is white noise with variance 1.

(a) With data up to time \(T\), calculate the \(k\)-step ahead optimal forecast of \(\hat{X}_{T,k}\) for all \(k \geq 1\). By giving a general formula relating \(\hat{X}_{T,k}\), \(k \geq 3\), to \(\hat{X}_{T,1}\) and \(\hat{X}_{T,2}\), determine the curve on which all these forecasts lie.

(b) Suppose now that \(T = 95\). Calculate numerically the forecasts \(\hat{X}_{95,k}\), \(k = 1, 2, 3\) and their mean squared prediction errors when the last five observations are \(X_{91} = 15.1, X_{92} = 15.8, X_{93} = 15.9, X_{94} = 15.2, X_{95} = 15.9\).

[You will need estimates for \(\epsilon_{94}\) and \(\epsilon_{95}\). Start by assuming \(\epsilon_{91} = \epsilon_{92} = 0\), then calculate \(\hat{\epsilon}_{93} = \epsilon_{93} = X_{93} - \hat{X}_{92,1}\), and so on, until \(\epsilon_{94}\) and \(\epsilon_{95}\) are obtained.]
13. Consider the state space model,

\[ X_t = S_t + v_t, \]
\[ S_t = S_{t-1} + w_t, \]

where \( X_t \) and \( S_t \) are both scalars, \( X_t \) is observed, \( S_t \) is unobserved, and \( \{v_t\}, \{w_t\} \) are Gaussian white noise sequences with variances \( V \) and \( W \) respectively. Write down the Kalman filtering equations for \( \hat{S}_t \) and \( P_t \).

Show that \( P_t \equiv P \) (independently of \( t \)) if and only if \( P^2 + PW = WV \), and show that in this case the Kalman filter for \( \hat{S}_t \) is equivalent to exponential smoothing.