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Time Series — Examples Sheet

This is the examples sheet for the M. Phil. course in Time Series. A copy can be found at: `http://www.statslab.cam.ac.uk/~rrw1/timeseries/`

Throughout, unless otherwise stated, the sequence $\{\epsilon_t\}$ *is white noise, variance* σ^2.

1. Find the Yule-Walker equations for the AR(2) process

$$X_t = \tfrac{1}{3}X_{t-1} + \tfrac{2}{9}X_{t-2} + \epsilon_t \,.$$

Hence show that it has autocorrelation function

$$\rho_k = \tfrac{16}{21}\left(\tfrac{2}{3}\right)^{|k|} + \tfrac{5}{21}\left(-\tfrac{1}{3}\right)^{|k|}, \quad k \in \mathbb{Z}\,.$$

2. Let $X_t = A\cos(\Omega t + U)$, where A is an arbitrary constant, Ω and U are independent random variables, Ω has distribution function F over $[0, \pi]$, and U is uniform over $[0, 2\pi]$. Find the autocorrelation function and spectral density function of $\{X_t\}$. Hence show that, for any positive definite set of covariances $\{\gamma_k\}$, there exists a process with autocovariances $\{\gamma_k\}$ such that every realization is a sine wave.

[Use the following definition: $\{\gamma_k\}$ are positive definite if there exists a nondecreasing function F such that $\gamma_k = \int_{-\pi}^{\pi} e^{ik\omega} dF(\omega)$.]

3. Find the spectral density function of the AR(2) process

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \epsilon_t \, .$$

What conditions on (ϕ_1, ϕ_2) are required for this process to be an indeterministic second order stationary? Sketch in the (ϕ_1, ϕ_2) plane the stationary region.

4. For a stationary process define the covariance generating function

$$g(z) = \sum_{k=-\infty}^{\infty} \gamma_k z^k, \quad |z| < 1\,.$$

Suppose $\{X_t\}$ satisfies $X = C(B)\epsilon$, that is, it has the Wold representation

$$X_t = \sum_{r=0}^{\infty} c_r \epsilon_{t-r}\,,$$

where $\{c_r\}$ are constants satisfying $\sum_0^\infty c_r^2 < \infty$ and $C(z) = \sum_{r=0}^\infty c_r z^r$. Show that

$$g(z) = C(z)C(z^{-1})\sigma^2\,.$$

Explain how this can be used to derive autocovariances for the ARMA(p,q) model. Hence show that for ARMA$(1,1)$, $\rho_2^2 = \rho_1\rho_3$. How might this fact be useful?

5. Consider the ARMA$(2,1)$ process defined as

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \epsilon_t + \theta_1 \epsilon_{t-1}\,.$$

Show that the coefficients of the Wold representation satisfy the difference equation

$$c_k = \phi_1 c_{k-1} + \phi_2 c_{k-2}, \quad k \geq 2\,,$$

and hence that

$$c_k = A z_1^{-k} + B z_2^{-k}\,,$$

where z_1 and z_2 are zeros of $\phi(z) = 1 - \phi_1 z - \phi_2 z^2$, and A and B are constants. Explain how in principle one could find A and B.

6. Suppose

$$Y_t = X_t + \epsilon_t, \qquad X_t = \alpha X_{t-1} + \eta_t \,,$$

where $\{\epsilon_t\}$ and $\{\eta_t\}$ are independent white noise sequences with common variance σ^2. Show that the spectral density function of $\{Y_t\}$ is

$$f_Y(\omega) = \frac{\sigma^2}{\pi} \left\{ \frac{2 - 2\alpha \cos\omega + \alpha^2}{1 - 2\alpha \cos\omega + \alpha^2} \right\} .$$

For what values of p, d, q is the autocovariance function of $\{Y_t\}$ identical to that of an ARIMA(p, d, q) process?

7. Suppose $X_1, \ldots, X_T$ are values of a time series. Prove that

$$\left\{ \hat{\gamma}_0 + 2 \sum_{k=1}^{T-1} \hat{\gamma}_k \right\} = 0 \,,$$

where $\hat{\gamma}_k$ is the usual estimator of the kth order autocovariance,

$$\hat{\gamma}_k = \frac{1}{T} \sum_{t=k+1}^{T} (X_t - \bar{X})(X_{t-k} - \bar{X}) \,.$$

Hint: Consider $0 = \sum_{t=1}^{T} (X_t - \bar{X})$.

Hence deduce that not all ordinates of the correlogram can have the same sign.

Suppose $f(\cdot)$ is the spectral density and $I(\cdot)$ the periodogram. Suppose f is continuous and $f(0) \neq 0$. Does $\mathbb{E} I(2\pi/T) \to f(0)$ as $T \to \infty$?

8. Suppose $I(\cdot)$ is the periodogram of $\epsilon_1, \ldots, \epsilon_T$, where these are i.i.d. $N(0,1)$ and $T = 2m+1$. Let ω_j, ω_k be two distinct Fourier frequencies, Show that $I(\omega_j)$ and $I(\omega_k)$ are independent random variables. What are their distributions?

If it is suspected that $\{\epsilon_t\}$ departs from white noise because of the presence of a single harmonic component at some unknown frequency ω a natural test statistic is the maximum periodogram ordinate

$$T = \max_{j=1,\ldots,m} I(\omega_j)\,.$$

Show that under the hypothesis that $\{\epsilon_t\}$ is white noise

$$P(T > t) = 1 - \left\{1 - \exp\left(-\pi t/\sigma^2\right)\right\}^m\,.$$

9. Complete this sketch of the fast Fourier transform. From data $X_0, \dots, X_T$, with $T = 2^M - 1$, we want to compute the 2^{M-1} ordinates of the periodogram

$$I(\omega_j) = \frac{1}{\pi T} \left| \sum_{t=0}^{T} X_t e^{it2\pi j/2^M} \right|^2, \quad j = 1, \dots, 2^{M-1}.$$

This requires order T multiplications for each j and so order T^2 multiplications in all. However,

$$\begin{aligned} \sum_{t=0,1,\dots,2^M-1} X_t e^{it2\pi j/2^M} &= \sum_{t=0,2,\dots,2^M-2} X_t e^{it2\pi j/2^M} + \sum_{t=1,3,\dots,2^M-1} X_t e^{it2\pi j/2^M} \\ &= \sum_{t=0,1,\dots,2^{M-1}-1} X_{2t} e^{i2t2\pi j/2^M} + \sum_{t=0,1,\dots,2^{M-1}-1} X_{2t+1} e^{i(2t+1)2\pi j/2^M} \\ &= \sum_{t=0,1,\dots,2^{M-1}-1} X_{2t} e^{it2\pi j/2^{M-1}} + e^{i2\pi j/2^M} \sum_{t=0,1,\dots,2^{M-1}-1} X_{2t+1} e^{it2\pi j/2^{M-1}}. \end{aligned}$$

Note that the value of either sum on the right hand side at $j = k$ is the complex conjugate of its value at $j = (2^{M-1} - k)$; so these sums need only be computed for $j = 1, \dots, 2^{M-2}$. Thus we have two sums, each of which is similar to the sum on the left hand side, but for a problem half as large. Suppose the computational effort required to work out each right hand side sum (for all 2^{M-2} values of j) is $\Theta(M-1)$. The sum on the left hand side is obtained (for all 2^{M-1} values of j) by combining the right hand sums, with further computational effort of order 2^{M-1}. Explain

$$\Theta(M) = a2^{M-1} + 2\Theta(M-1).$$

Hence deduce that $I(\cdot)$ can be computed (by the FFT) in time $T \log_2 T$.

10. Suppose we have the ARMA$(1,1)$ process

$$X_t = \phi X_{t-1} + \epsilon_t + \theta \epsilon_{t-1} \,,$$

with $|\phi| < 1$, $|\theta| < 1$, $\phi + \theta \neq 0$, observed up to time T, and we want to calculate k-step ahead forecasts $\hat{X}_{T,k}$, $k \geq 1$.

Derive a recursive formula to calculate $\hat{X}_{T,k}$ for $k = 1$ and $k = 2$.

11. Consider the stationary scalar-valued process $\{X_t\}$ generated by the moving average, $X_t = \epsilon_t - \theta\epsilon_{t-1}$.

Determine the linear least-square predictor of X_t, in terms of $X_{t-1}, X_{t-2}, \dots$.

12. Consider the ARIMA$(0, 2, 2)$ model

$$(I - B)^2 X = (I - 0.81B + 0.38B^2)\epsilon$$

where $\{\epsilon_t\}$ is white noise with variance 1.

(a) With data up to time T, calculate the k-step ahead optimal forecast of $\hat{X}_{T,k}$ for all $k \geq 1$. By giving a general formula relating $\hat{X}_{T,k}$, $k \geq 3$, to $\hat{X}_{T,1}$ and $\hat{X}_{T,2}$, determine the curve on which all these forecasts lie.

(b) Suppose now that $T = 95$. Calculate numerically the forecasts $\hat{X}_{95,k}$, $k = 1, 2, 3$ and their mean squared prediction errors when the last five observations are $X_{91} = 15.1$, $X_{92} = 15.8$, $X_{93} = 15.9$, $X_{94} = 15.2$, $X_{95} = 15.9$.

[You will need estimates for ϵ_{94} and ϵ_{95}. Start by assuming $\epsilon_{91} = \epsilon_{92} = 0$, then calculate $\hat{\epsilon}_{93} = \epsilon_{93} = X_{93} - \hat{X}_{92,1}$, and so on, until ϵ_{94} and ϵ_{95} are obtained.]

13. Consider the state space model,

$$X_t = S_t + v_t,$$
$$S_t = S_{t-1} + w_t\,,$$

where X_t and S_t are both scalars, X_t is observed, S_t is unobserved, and $\{v_t\}$, $\{w_t\}$ are Gaussian white noise sequences with variances V and W respectively. Write down the Kalman filtering equations for $\hat{S}_t$ and P_t.

Show that $P_t \equiv P$ (independently of t) if and only if $P^2 + PW = WV$, and show that in this case the Kalman filter for $\hat{S}_t$ is equivalent to exponential smoothing.