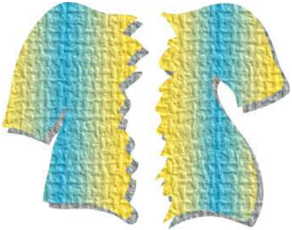
THE DISPUTED GARMENT PROBLEM: THE MATHEMATICS OF BARGAINING & ARBITRATION



Richard Weber

Nicky Shaw Public Understanding of Mathematics Lecture

7 February, 2008

The Disputed Garment Problem

The **Babylonian Talmud** is the compilation of ancient law and tradition set down during the first five centuries A.D. which serves as the basis of Jewish religious, criminal and civil law. One problem discussed in the Talmud is the disputed garment problem.

> **"Two hold a garment; one claims it all, the other claims half. Then one is awarded $\frac{3}{4}$ and the other $\frac{1}{4}$."**

The idea is that half of the garment is not in dispute and can be awarded to the one who claims the whole garment. The other half of the garment is in dispute and should be split equally.
Thus one gets $\frac{1}{2} + \frac{1}{4}$ and the other gets $\frac{1}{4}$.

Sharing the Cost of a Runway

Suppose three airplanes share a runway.

- Plane 1 requires 1 km to land.
- Plane 2 requires 2 km to land.
- Plane 3 requires 3 km to land.

So a runway of 3 km must be built.

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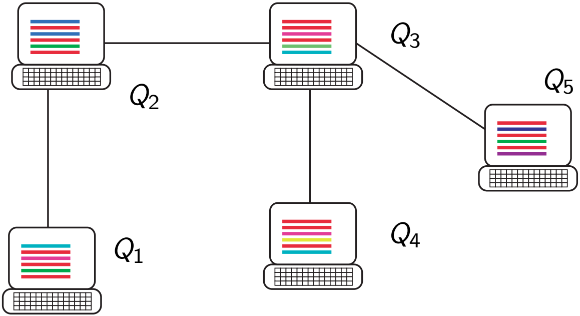
What proportion of the building cost should each plane pay?

Sharing Files

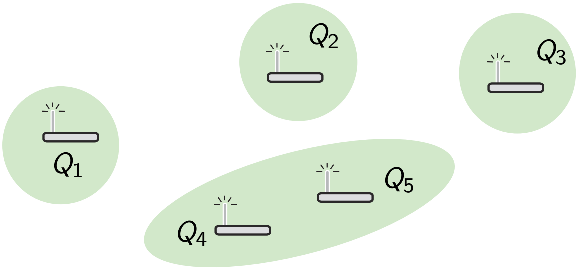
A file sharing system Peers contribute files to a shared library of files which they can access over the Internet.



Peer i shares Q_i files.
The benefit to peer j is $\theta_j u(Q_1 + \cdots + Q_5)$.

Sharing WLANS

A sharing of wireless LANS system Peers share their wireless Local Area Networks so that they can enjoy Internet access via one another's networks whenever they wander away from their home locations.



Peer i makes his WLAN available for a fraction Q_i of the time. The benefit to peer j is $\theta_j u(Q_1 + \cdots + Q_5)$.

Example: A Bridge

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Mathematical Bridge, Queens' College, Cambridge

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What is the best fee mechanism, $p_1(\theta_1, \theta_2)$ and $p_2(\theta_1, \theta_2)$?

Tractate Kethuboth

The Baylonian Talmud also gives instructions about dividing estates.

MISHNAH 93. IF A MAN WHO WAS MARRIED TO THREE WIVES DIED, AND THE KETHUBAH OF ONE WAS A MANEH, OF THE OTHER TWO HUNDRED ZUZ, AND OF THE THIRD THREE HUNDRED ZUZ AND THE ESTATE [WAS WORTH] ONLY ONE MANEH [THE SUM] IS DIVIDED EQUALLY. IF THE ESTATE [WAS WORTH] TWO HUNDRED ZUZ [THE CLAIMANT] OF THE MANEH RECEIVES FIFTY ZUZ [AND THE CLAIMANTS RESPECTIVELY] OF THE TWO HUNDRED AND THE THREE HUNDRED ZUZ [RECEIVE EACH] THREE GOLD DENARII. IF THE ESTATE [WAS WORTH] THREE HUNDRED ZUZ, [THE CLAIMANT] OF THE MANEH RECEIVES FIFTY ZUZ AND [THE CLAIMANT] OF THE TWO HUNDRED ZUZ [RECEIVES] A MANEH WHILE [THE CLAIMANT] OF THE THREE HUNDRED ZUZ [RECEIVES] SIX GOLD DENARII.

The Marriage Contract Problem

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What happens when the man dies with less than 600 zuz?

The Marriage Contract Problem

The Talmud gives recommendations.

	Debt		
Estate	100	200	300
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100: equal division

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Estate	100	200	300
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200	50	75	75
300	50	100	200

100: equal division
300: proportional division
200: ?

The Bankruptcy Game

Two creditors have claims for \$30 million and \$70 million against a bankrupt company. The company only has \$60 million.

The players must reach an agreement about how to divide the money between them, i.e., to choose a_1, a_2, such that

$$\begin{gathered}\text{Creditor 1 gets } a_1;\\ \text{Creditor 2 gets } a_2;\\ \text{with } a_1 + a_2 \leq 60.\end{gathered}$$

Players are equally powerful and have equally good lawyers. Once all the arguments have been made and 'the dust has settled' how much money should each get?

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What would be a 'fair' division of the money?

Possible Lawyer's Arguments

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(d) The disputed garment principle. Creditor 2 should be awarded at least 30, since this is what would be left for him if he first paid Creditor 1's entire claim,

$$30 = 60 - 30 \text{ (Creditor 1's entire claim).}$$

30 is in dispute, and is split equally, giving $\mathbf{(15, 45)}$.

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$$30 = 60 - 30 \text{ (Creditor 1's entire claim).}$$

30 is in dispute, and is split equally, giving $\mathbf{(15, 45)}$.
Each suffers the same loss compared to what he would get if he were the only creditor, i.e., $\mathbf{(30, 60) - (15, 45) = (15, 15)}$.

John Nash, 1928–

Equilibrium Points in *N*-person Games, *Proceedings of the National Academy of Sciences* 36 (1950).

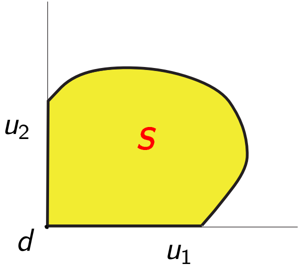
'The Bargaining Problem', *Econometrica* 18 (1950).

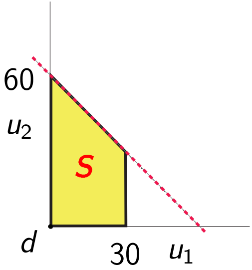
'Two-person Cooperative Games', *Econometrica* 21 (1953).

Nobel Prize in Economics (1994)

Nash's Bargaining Game

We can represent the bargaining game in the following picture.





Two players attempt to agree on a point $u = (u_1, u_2)$, in the set S.
If they agree on $u = (u_1, u_2)$ their 'happinesses' are u_1 and u_2 respectively.
If cannot agree they get $d = (d_1, d_2)$ (the 'disagreement point').

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2. **Symmetry**. If their claims are exactly the same then they should split the money in half.
3. **Independence of Irrelevant Alternatives**.
   You and I are deciding upon a pizza to order and share. We decide on a pepperoni pizza, with no anchovies. Just as we are about to order, the waiter tells us that the restaurant is out of anchovies. Knowing this, it would now be silly to decide to switch to having a mushroom pizza.
   The fact that anchovies are not available is irrelevant, since we did not want them anyway.

4. **Scale invariance**. If we rescale units then the solution should not change.

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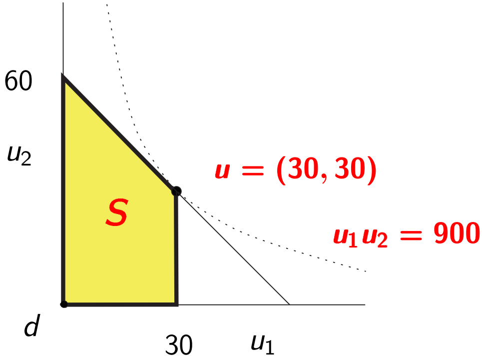
$$u_1' = a_1 + b_1 u_1$$
$$u_2' = a_2 + b_2 u_2$$

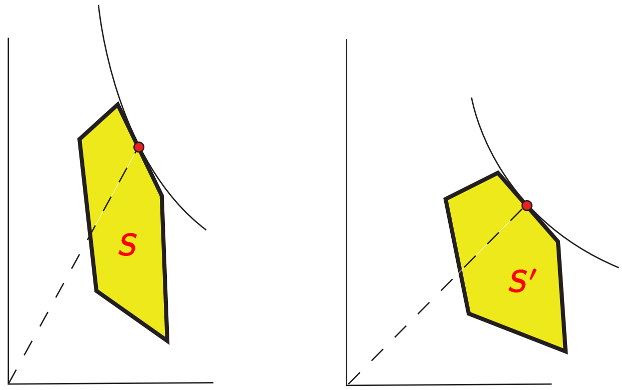
and $(\bar{u}_1, \bar{u}_2)$ is the solution to the game played in bargaining set S and with disagreement point $d = (d_1, d_2)$, then $(a_1 + b_1 \bar{u}_1, a_2 + b_2 \bar{u}_2)$ is the solution to that game played in bargaining set $S' = \{(v_1, v_2) \ : \ v_i = a_i + b_i u_i \,, (u_1, u_2) \in S\}$ and with disagreement point $d' = (a_1 + b_1 d_1, a_2 + b_2 d_2)$.

Nash Solution of the Bargaining Game

Theorem 1 *There is one and only one way to satisfy the Nash bargaining axioms. It is to choose the point in* $\boldsymbol{S}$ *which maximizes* $(u_1 - d_1)(u_2 - d_2)$.

Creditors 1 and 2 have valid claims for 30 and 70. But there is only 60 to share. The Nash bargaining solution is $u = (30, 30)$.

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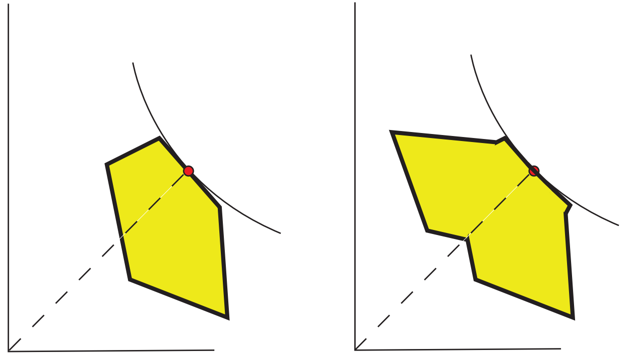

Objections and Counterobjections

Player 1 has an objection to (u_1, u_2) if there is probability p_1 that he can force Player 2 to accept some (v_1, v_2) (otherwise negotiations breakdown) and

$$p_1 v_1 + (1 - p_1) d_1 \geq u_1 \iff p_1 \geq \frac{u_1 - d_1}{v_1 - d_1}$$

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Player 2 has a valid counterobjection if for some $p_2 < p_1$,

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Suppose every objection to u has a valid counterobjection.
This requires that for all (v_1, v_2),

$$\frac{u_1 - d_1}{v_1 - d_1} \geq p_2 \geq \frac{v_2 - d_2}{u_2 - d_2}$$
$$\implies (u_1 - d_1)(u_2 - d_2) \geq (v_1 - d_1)(v_2 - d_2)$$

Back to the Marriage Contract Problem

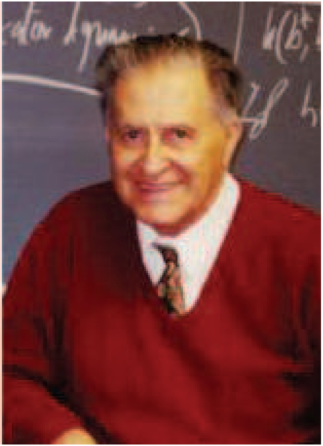
Recall that the Talmud recommends:

	Debt		
Estate	100	200	300
100	33.33	33.33	33.33
200	50	75	75
300	50	100	150

This baffled scholars for two millennia. In 1985, it was recognised that the Talmud anticipates the modern game theory.

The Talmud's solution is equivalent to the nucelolus of an appropriately defined cooperative game. The nucleolus is defined in terms of objections and counterobjections.

Robert Aumann and Michael Maschler

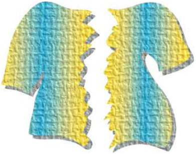
 

Game-Theoretic Analysis of a Bankruptcy Problem from the Talmud, *Journal of Economic Theory* (1985).

Consistency

The consistency principle. If the division amongst n players gives players i and j amounts a_i and a_j, then these are the same amounts they would get in the solution to a problem in which $a_i + a_j$ is to be divided between i and j.

Recall the disputed garment principle:



"Two hold a garment; one claims it all, the other claims half. Then one is awarded $\frac{3}{4}$ and the other $\frac{1}{4}$."

Consistency

Suppose the man leaves 200 zuz. Wives 1 and 2 claim 100 and 200, respectively, and Talmud awards them 50 and 75 zuz.

If they were dividing their total of 50+75=125 zuz, they would say that 25 in not in dispute (since wife 1 is only claiming 100). So that 25 should go to wife 2 and the remaining 100 should be shared equally, giving the allocation $(50, 75)$.

Consistency

Theorem 2 *The Talmud solution is the unique solution that is consistent with the disputed garment principle.*

So if everyone likes the disputed garment principle, then the Talmud solution avoids the possibility that any two wives will disagree about how what they have has been split between them.

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Good Aspects of the Nash solution

The Nash bargaining solution extends to $n > 2$ players.
The solution is to maximize $(u_1 - d_1) \cdots (u_n - d_n)$ over $u \in S$.

This has two good properties:

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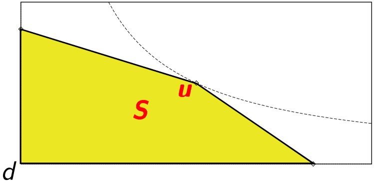
- Consistency.
- Proportional fairness.
  Suppose $\bar{u}$ is the Nash solution and u is any other solution. Then
  $$\sum_{i=1}^{n} \frac{u_i - \bar{u}_i}{\bar{u}_i} \leq 0.$$
  That is, for any move away from the Nash solution the sum of the percentage changes in the utilities is negative.

A Problem with the Nash solution

The Nash bargaining solution does not have the property of Monotonicity.

I.e., if $\bar{u}$ and $\bar{u}'$ are the solutions for bargaining sets S and S' respectively, and S is contained in S', then it is not necessarily the case that $\bar{u}'_i \geq \bar{u}_i$ for both $i = 1$ and $i = 2$.



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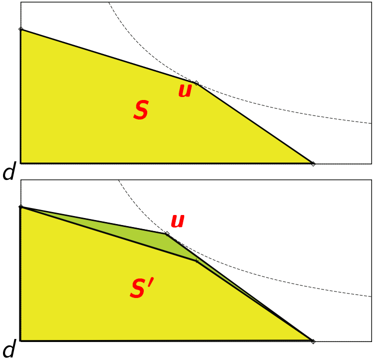

Arrow's Impossibility Theorem

Each voter has a preference rank amongst candidates. We would like to compute a preference ranking for society, taken as a whole. This preference rank, $\succ$, should satisfy properties of

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Impossibility Theorem. *It is not possible to satisfy all the above.*

Coalitions in Provision of Telecommunications Links

The savings in the costs of providing links for Australia, Canada, France, Japan, UK and USA can be defined as

$v(T) = \text{(cost separate)} - \text{(cost in coalition } \boldsymbol{T}\text{)}$.

subset S	separate	coalition	$\boldsymbol{v(T)}$	saving (%)
J UK USA	13895	11134	2761	20
A UK USA	12610	10406	2204	17
F J USA	6904	5609	1295	19
A F USA	5600	4801	799	14
C J UK	3995	3199	796	20
A C UK	3869	3127	742	19
A J UK USA	18558	13573	4985	27
F J UK USA	20248	16733	3515	17
A F J USA	18847	15982	2865	15
C J UK USA	15860	13044	2816	18
A F J UK USA	24990	19188	5802	23
A C J UK USA	20667	15570	5097	25

Arbitration

Consider a set of n players, $N = \{1, 2, \ldots, n\}$. The value they obtain by cooperation in some activity is $v(N)$. If a subset, T, cooperate as a coalition they get $v(T)$.

E.g., in a bankruptcy, in which creditor i is owed c_i and the estate has value E, we could have:

$$v(T) = \max\left\{E - \sum_{i \notin T} c_i, 0\right\}.$$

This is the amount left for those in coalition T to share after they have paid off everyone not in T.

For disjoint sets T and U,

$$v(T \cup U) \geq v(T) + v(U).$$

The Job of an Arbitrator

The job of an arbitrator is to 'divide the spoils' of the grand coalition, e.g., to make an award $x_1, \dots, x_n$, (called an **imputation**), to players $1, \dots, n$, such that

$$x_1 + \cdots + x_n = v(N),$$

and no one can object.

Arbitration is accomplished by a function ϕ such that

$$(x_1, \dots, x_n) = \big(\phi_1(N), \dots, \phi_n(N)\big).$$

$\phi(\cdot)$ also encapsulates the way the arbitrator would divide $v(T)$ amongst the members of any subset $T \subset N$.

Objections and Counterobjections

If $\phi_j(N) > \phi_j(N - \{i\})$, then player i might threaten player j, "*give me more or I will leave the coalition and you will lose*."

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Player j has a valid counterobjection if he can point out that if he leaves the coalition then i loses just as much.

Objections and Counterobjections

If $\phi_j(N) > \phi_j(N - \{i\})$, then player i might threaten player j, "*give me more or I will leave the coalition and you will lose*."

Player j has a valid counterobjection if he can point out that if he leaves the coalition then i loses just as much.

$$\phi_i(N) - \phi_i(N - \{j\}) \geq \phi_j(N) - \phi_j(N - \{i\})$$

Objections and Counterobjections

If $\phi_j(N) < \phi_j(N - \{i\})$, player j might threaten player i, "*give me more or I will convince the others to exclude you and I will be better off*."

Player i has a valid counterobjection if he can point out that if he gets the others to exclude j then i will be better off by at least as much.
If the arbitrator is to make sure that every such objection has a counterobjection, he must ensure

$$\phi_i(N) - \phi_i(N - \{j\}) = \phi_j(N) - \phi_j(N - \{i\}).$$

Shapley Value

So if each objection has a counterobjection, we require

$$\boldsymbol{\phi_i(N) - \phi_i(N - \{j\}) = \phi_j(N) - \phi_j(N - \{i\})}.$$

Only one function $\phi(\cdot)$ does this: the Shapley value function. Its value for player i is the expected amount he brings to the coalition when the coalition is formed in random order.

Shapley Values for the Bankruptcy Game

$$v(T) = \max\left\{E - \sum_{i \notin T} c_i, 0\right\}.$$

$c_1 = 30,\ c_2 = 70,\ E = 60.$

$v(\{1\}) = 0,\ v(\{2\}) = 30,\ v(\{1,2\}) = 60.$

Player 1 brings 0 if he comes first and 30 if he comes second.

$$\phi_1(N) = \frac{1}{2}(\ \ 0 + 30) = 15$$

Player 2 brings 30 if he comes first and 60 if he comes second.

$$\phi_2(N) = \frac{1}{2}(30 + 60) = 45$$

Sharing the Cost of a Runway

The Shapley value has been used for cost sharing.
Suppose three airplanes share a runway. The planes require 1, 2 and 3 km to land, respectively. So a runway of 3km must be built. What should each pay?

	adds cost		
order	1	2	3
1,2,3	1	1	1
1,3,2	1	0	2
2,1,3	0	2	1
2,3,1	0	2	1
3,1,2	0	0	3
3,2,1	0	0	3
Total	2	5	11

So they should pay for $\mathbf{2/6}$, $\mathbf{5/6}$ and $\mathbf{11/6}$ km, respectively.

Shapley Value

The Shapley value computes each player's bargaining power in terms of the value he contributes. Persons who contribute more should receive a higher percentage of the benefits.
The Shapley value is also the only value which satisfies four axioms, namely,

- treatment of all players is symmetric.
- non-contributors receive nothing. We say i is a non-contributor if $v(T + \{i\}) = v(T)$ for all subsets T.
- there is no division that makes everyone better off.
- $\phi_i^{v+v'}(N) = \phi_i^{v}(N) + \phi_i^{v'}(N)$. E.g., participant i's cost-share of a runway and terminal is his cost-share of the runway, plus his cost-share of the terminal.

Political Power

The Shapley value has also been used to assess political power. In 1964 the Board of Supervisor of Nassau County operated by weighted voting. There were six members, with weights of $\{31, 31, 28, 21, 2, 2\}$.
Majority voting operates, so for $T \subseteq \{1, 2, 3, 4, 5, 6\}$, let

$$v(T) = \begin{cases} 1 \text{ if } T \text{ has a total weight 58 or more;} \\ 0 \text{ otherwise.} \end{cases}$$

The Shapley values are $\big(.33, .33, .33, 0, 0, 0\big)$.

This shows that 3 members are totally without influence!

The Nucleolus

A final characterization of the Talmud solution is the following. x is called an imputation (a 'division of the spoils') if

$$\text{(a) } \sum_{i \in N} x_i = v(N)\,, \text{ and (b) } x_i \geq v(\{i\}) \text{ for all } i.$$

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Let

$$U(x, T) = v(T) - \sum_{i \in T} x_i$$

This is the 'unhappiness' felt by a subset of players T.

The Nucleolus

Suppose a subset of players, T, has an objection to imputation x because T would be less unhappy if the imputation were changed to y.

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Suppose that all possible objections to x have counterobjections.

There is only one such x and it is called the **nucleolus**.

The Nucleolus of the Estate Division Problem

Recall, we have an estate of value E and claims $c_1, \ldots, c_n$. Let T be a subset of the set of numbers $\{1, 2, \ldots, n\}$.
Suppose we again take

$$v(T) = \max\left\{E - \sum_{i \notin T} c_i, 0\right\}$$

i.e., the participants in T get to share the amount of money left (if any) once everyone not in T has had their claims paid in full.

Theorem 3 *The Talmud's solution is the nucleolus of the 'coalitional game' with the above $v(\cdot)$.*

Nucleolus for Estate of 200

Estate is 200. Wives claim 100, 200, 300. Nucleolus is $x^* = (50, 75, 75)$.

T	$v(T)$	$U(x, T)$	$U(x^*, T)$
$\{1\}$	0	$0 - x_1$	-50
$\{2\}$	0	$0 - x_2$	-75
$\{3\}$	0	$0 - x_3$	-75
$\{1, 2\}$	0	$0 - x_1 - x_2$	-125
$\{1, 3\}$	0	$0 - x_1 - x_3$	-125
$\{2, 3\}$	100	$100 - x_2 - x_3$	-50
$\{1, 2, 3\}$	200	$200 - x_1 - x_2 - x_3$	0

The nucleolus ‘lexicographically’ minimizes the ordered unhappinesses. E.g. $(0, -50, -50, -75, -75, -125, -125)$.
The unhappiness of the ‘most unhappy’ subset of players is minimized. Subject to not reducing that subset’s happiness, the unhappiness of the next most unhappy subset of players in minimized. And so on. This is also called ‘max-min fairness’.

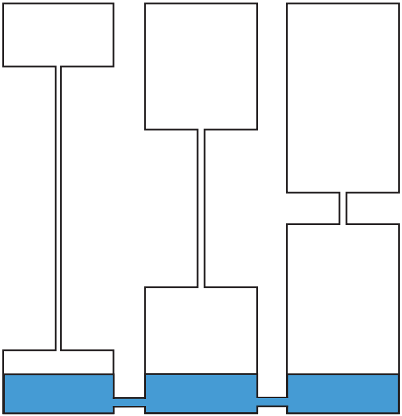
Nucleolus for Estate of 400

Estate is 400. Wives claim 100, 200, 300. Nucleolus is $x^* = (50, 125, 225)$.

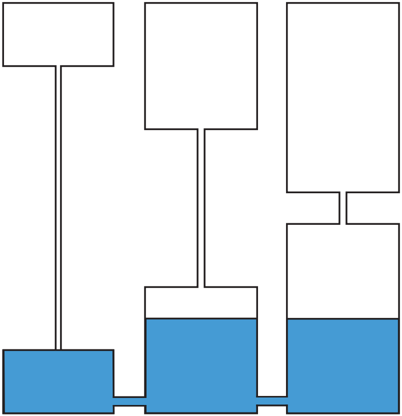
T	$v(T)$	$U(x, T)$	$U(x^*, T)$
$\{1\}$	0	$0 - x_1$	-50
$\{2\}$	0	$0 - x_2$	-125
$\{3\}$	100	$100 - x_3$	-125
$\{1,2\}$	100	$100 - x_1 - x_2$	-75
$\{1,3\}$	200	$200 - x_1 - x_3$	-75
$\{2,3\}$	300	$300 - x_2 - x_3$	-50
$\{1,2,3\}$	400	$400 - x_1 - x_2 - x_3$	0

The nucleolus 'lexicographically' minimizes the ordered unhappinesses. E.g. $(0, -50, -50, -75, -75, -125, -125)$.

Connected vessels



Connected vessels



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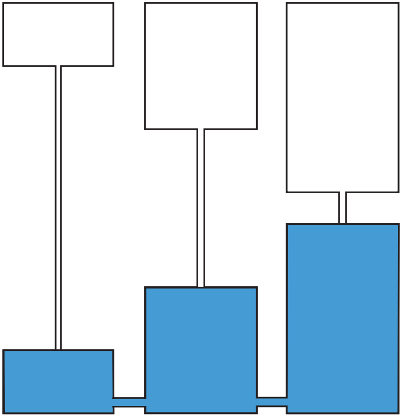

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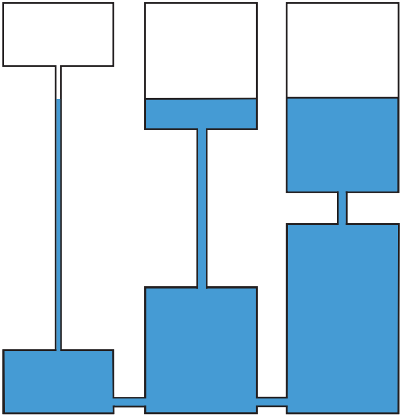

Optimal fees for the bridge

The bridge costs 1 to build and $\theta_1, \theta_2 \sim U[0,1]$.
Best possible create social welfare of $1/6 = 0.166\dot{6}$.

A 'second-best' builds the bridge only if $\theta_1 + \theta_2 \geq 1.25$ and has social welfare $9/64 = 0.140625$.

Agent 1 pays fee of

$$p_1(\theta_1, \theta_2) = \left(\frac{1}{3}(\theta_1 - \theta_2) + \frac{1}{2}\right) 1_{\{\theta_1 + \theta_2 \geq 1.25\}}\,.$$

Conclusions

- There are many 'solution concepts'. E.g., Nash, Kalai-Smorodinski, Shapley, Talmud, consistency, proportional fairness, max-min fairness, etc.

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Conclusions

- There are many ‘solution concepts’. E.g., Nash, Kalai-Smorodinski, Shapley, Talmud, consistency, proportional fairness, max-min fairness, etc.
- Any one solution concept will usually violate the axioms associated with some other solution concept.
  If axioms are meant to represent intuition, then counter-intuitive examples are inevitable.
- A ‘perfect’ solution to a bargaining, arbitration or voting problem is unattainable.
  One must choose a solution concept on the basis of what properties one likes and what counter-intuitive examples one wishes to avoid.