# Mechanism Design in Shared Infrastructures

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# Motivation: Shared Infrastructures

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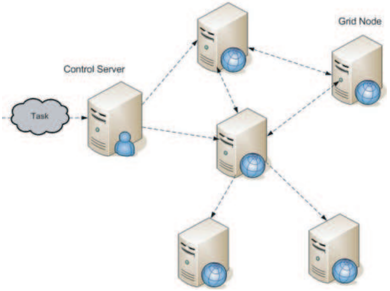
**Grid computing**



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Seeks to understand how markets can be designed to exploit benefits of Grid computing.

# A market solution

One possible approach is to form a market for computation. In this market providers (sellers) and consumers (buyers) of computing resources go to trade.

For instance, an organization might go to the market and say that it needs 10 virtual machines of a certain type for 8 hours and state that the maximum price it is willing to pay is 100 euros. This corresponds to a 'bid' in this market. Similarly, an organization can post in the market its excess computing resources with an 'ask' of the minimum price at which it is willing to sell. The market matches the asks and bids, just as in the stock market, and allocates resources accordingly.

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Our approach differs.
We provide rules for bukilding and sharing a resource pool.

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We wish to

- eliminate the free-rider problem;
- incentivize agents to truthfully reveal private information.

**Key observation**: agents will adopt strategies that depend on how a system is operated.

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- $u_i(\cdot)$ is pubic knowledge, but only agent $i$ knows $\theta_i$.
- $\omega$ is to be chosen on the basis of $S$ and declared $\theta_i$.

# Agents pay for operating cost

Suppose agent $i$ is charged a fee $p_i(S, \theta)$.

Fees are used to cover a daily operating cost, $c$. So we require

$$E_{S,\theta}\left[p_1(S, \theta) + \cdots + p_n(S, \theta)\right] \geq c\,.$$

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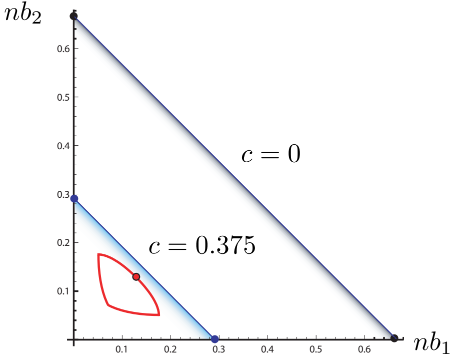
In some situations we may take the fee as money.

In others we may wish to take the fee as a contribution to the pool of resources that is available in the infrastructure.

# The efficient frontier

We wish to find Pareto optimal points of the vector

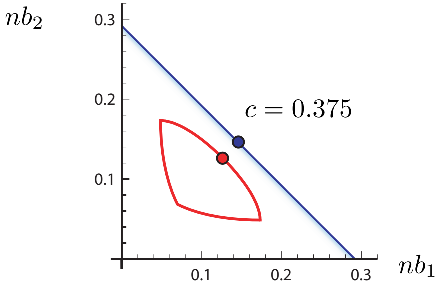
$$(nb_1, \ldots, nb_n)$$



# Maximum social welfare

Suppose we wish to find the particular point that maximizes

$$nb_1 + \cdots + nb_n = E_{S,\theta}\left[\theta_1 u_1(\omega(S)) + \cdots + \theta_n u_n(\omega(S))\right] - c$$




We call this the ‘social welfare’.

# Our infrastructure optimization problem

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Do the two things above, as function of declared $\theta_i$, so that:

1. Users find it in their best interest to truthfully reveal their $\theta_i$.
2. Users see positive expected net benefit from participation.
3. Expected total fees cover the daily running cost, say $c$.
4. Expected social welfare (total net benefit) is maximized

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Fees should incentivize users to truthfully reveal $\theta_1$, $\theta_2$, with

$$p_1(\theta_1, \theta_2) + p_2(\theta_1, \theta_2) = 1 \text{ or } 0\,, \text{ as bridge is built or not built}\,.$$

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- whether or not the bridge is built;
- what contributions the users should make towards its cost;
- who gets to use the bridge on those days that both users say that they wish to do so.

# Motivation

Similarly, in grid computing:

- how do we incentivize agents to participate and contribute computational resource?
- what size of computational resource will be installed?
- what contributions should agents make towards its cost — or what amounts of resource should they be willing to contribute?
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Are auction and mechanism design theory appropriate? And under what assumptions on our model are these applicable?

What is fundamentally new in this problem?

Can we describe optimal policies?

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2. The facility does not exist; its size will be the sum of participants' contributions.

How to share resources and recover costs?

- Easy when we know utilities of participants.
- In practice agents' utilities are private information.
  We must design the system to operate well, under the constraint that each agent will reveal information in a manner that is to his best advantage.

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- Agent $i$ suffers delay cost, so his net benefit is, say,

$$nb_i = \lambda_i r - \theta_i \lambda_i \frac{1}{\sum_k y_k - \sum_k \lambda_k} - y_i \, .$$

$\theta_i$ is private information of agent $i$, but it has an *a priori* distribution that is public information.

# Optimal queue scheduling

Instead of declaring contributions they are willing to make, we can imagine that agents (equivalently) declare their $\theta_i$.

Suppose $\theta_1 < \theta_2 < \cdots < \theta_n$.

As a function of these declarations we take contributions of the form $y(\theta_i)$ from some subset of agents $i = 1, \ldots, j$ (a set with smallest $\theta_i$).

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Under this scheme, an agent with too great a $\theta_i$ will find unprofitable to consider participating.

$y_i(\theta_i)$ is increasing in $\theta_i$, and is determined by an incentive compatibility condition.

# A simple mathematical example

Consider a simple infrastructure shared by just 2 participants, both present on all days.
On day $t$, agent $i$ has utility for resource of $\theta_{i,t}u(x)$, where $\theta_{i,t}$ is known to be distributed $U[0,1]$.
The infrastructure is described by a single resource parameterized by a number (such as computing cycles); so operating methods are:

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Suppose $u(x) = x$. Focus on one day; let $\theta_{i,t} = \theta_i$.

$$E_{\theta_1,\theta_2}\left[\max_{x_1,x_2}\{\theta_1 u(x_1) + \theta_2 u(x_2)\}\right] = E\left[\max\{\theta_1, \theta_2\}\right] = \tfrac{2}{3}$$

We call this the 'first best'.

# Second-best solution

A ‘second-best’ mechanism can be constructed as follows.

If agent $i$ declares $\theta_i$ then he is charged a fee

$$p(\theta_i) = \begin{cases} (1/2)(\theta_i^2 + \theta_0^2)\,, & \theta_i \geq \theta_0 \\ 0\,, & \theta_i < \theta_0 \end{cases}$$

He obtains $x_i = 1$ if $\theta_i = \max\{\theta_1, \theta_2\} > \bar{\theta}_0$.

Note that the resource is given wholly to one agent.

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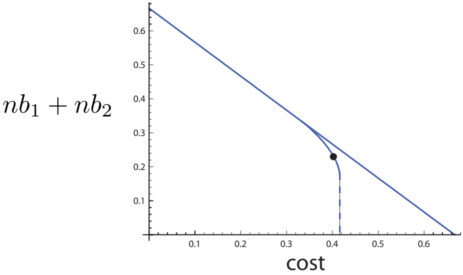
- Choosing $\theta_0$ s.t. the above equals $c$, maximizes social welfare:

$$E\left[\sum_{i=1}^{2} \theta_i u(x_i) - p(\theta_i)\right]$$

subject to covering cost $c$.

# Second-best versus first-best




Expected social welfare as a function of $c$, compared to first-best.

For $c \in [0.333, 0.416]$ the second-best falls short of the first-best. There is no way to cover a cost greater than $\frac{5}{12} = 0.416$.

# Other second-best mechanisms

Other mechanisms also work.

(a)

$$p_1(\theta_1, \theta_2) = \tfrac{1}{2}c + \tfrac{1}{2}(\theta_1^2 + \theta_0^2)1_{\{\theta_1 > \theta_0\}} - \tfrac{1}{2}(\theta_2^2 + \theta_0^2)1_{\{\theta_2 > \theta_0\}}$$

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There is 'ex-post' cost-covering.

(b)

$$p_1(\theta_1, \theta_2) = \max(\theta_0, \theta_2)1_{\{\theta_1 > \max(\theta_0, \theta_2)\}}$$

There is ex-post incentive compatibility and rationality.

# A model with true sharing

Suppose $u(x) = \sqrt{x}$. The resource is shared differently.
The optimal policy is found by solving a Lagrangian dual problem

$$\min_{\lambda \geq 0} \left\{ E_{\theta_1, \theta_2} \left[ \max_{\substack{x_1, x_2 \geq 0 \\ x_1 + x_2 \leq 1}} \sum_{i=1}^{2} h_\lambda(\theta_i) u(x_i) \right] - (1+\lambda)c \right\}.$$

where $h(\theta_i) = (\theta_i + \lambda(2\theta_i - 1))$ and

$$x_i(\theta_1, \theta_2) = \frac{h_\lambda(\theta_i)^2}{\sum_{j=1}^{2} h_\lambda(\theta_j)^2}$$

Fees increase with $\lambda$.
Social welfare decreases with $\lambda$, but is maximal subject to the constraint of covering the cost.

# The role of operating policy

The resource is not allocated in the 'most efficient' way.

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This is one of our most important lessons:

To optimally incentivize participation in shared infrastructures, and make the most of the resources available, one should appreciate that both (i) fee structure, **and** (ii) operating methods, must both play a part in providing the correct incentives to users.

# Building a facility from scratch

A different model: facility of size $Q$, costing $c(Q) = Q$ (per slot), is formed by initial contributions of agents.
These are incentivized to contribute because their contribution will affect the amount of resources they will get at run time.
Probably a good model for virtual Grid infrastructures.

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- System planner posts how he will compute agents' contributions and the $x_i(\boldsymbol{\theta}, S)$ as functions of the $\theta_i$s that they declare.
- Agents declare $\theta_i$s and system runs according to posted policy.

# Analysis of proportional sharing

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Consider 4 possible sharing disciplines:

- Acting alone: $x_i^{\{i\}} = x_i^{\{1,2\}} = q_i$.

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- Acting alone: $x_i^{\{i\}} = x_i^{\{1,2\}} = q_i$.
- Equal sharing: $x_i^{\{i\}} = q_1 + q_2$ and $x_i^{\{1,2\}} = \frac{1}{2}(q_1 + q_2)$.

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Consider simple case of 2 identical agents $\theta_1 = \theta_2 = 1$, $c(Q) = Q$.

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# Results for $\alpha_i = \alpha = 0.8$, $u(x) = 10 - 1/x$

| scheme | social welfare | values of $q_1, q_2$ |
|---|---|---|
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| | 6.21115 | 0.894427 |
| Equal sharing | $r\alpha - \frac{3}{2}\sqrt{\alpha(1+\alpha)}$ | $\frac{1}{2}\sqrt{\alpha(1+\alpha)}$ |
| $s = 0$ | 6.2 | 0.6 |
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How do these results generalize?

# The exact solution

Define $g_i(\theta_i) = \theta_i - (1 - F_i(\theta_i))/f_i(\theta_i)$

E.g., $g(\theta_i) = 2\theta_i - 1$ when $F_i$ is $U[0,1]$.

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There is a $\lambda \geq 0$, such that for all $S$ the optimal way to share resource amongst a set of active agents $S$ is to maximize

$$\textstyle\sum_{i \in S} (\theta_i + \lambda g(\theta_i)) u(x_i(\theta, S)) \,, \tag{1}$$

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Note $g(\theta_i)$ is increasing in $\theta_i$, but $E[g(\theta_i)] = 0$.
So an agent who declares a greater $\theta_i$ is receives more than a market allocation would give him when sharing the resource.

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It turns out that the solution of the Mechanism Design problem implies a simple 'effective bandwidth' tariff for type $i$ agents:

- System guarantees (with prob (1-$\epsilon$)) resource $y$ for a contribution of $\alpha_i y$ ($\alpha_i(1+\epsilon)y$).
- Agent $i$ indirectly declares his $\theta_i$ by selecting $y$ to maximize $\max_y\{\theta_i u(y) - \alpha_i y\}$.
- No information on $F_i$ required!

# Declaring activity frequencies

Now the $\alpha_i$ are private information, i.i.d. uniform on $[0, 1]$, and $\theta_{i,t} = \theta_i = 1$. Sensible if accounting of activity is costly.

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The facility is built from agent contributions.
We wish to compute the set of optimal tariffs $q(\omega), x(\omega)$ parametrized by $\omega$ the 'type' of the agent, where an agent that contributes $q(\omega)$ gets $x(\omega)$ when he is 'on'.

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An agent maximizes his net benefit $f(\alpha)$, where

$$f(\alpha) = \max\left\{ \max_{\omega} \big[ \alpha u(x(\omega)) - q(\omega) \big], 0 \right\}.$$

So need $d[\alpha u(x(\omega)) - q(\omega)]/d\omega|_{\omega=\alpha} = \alpha u'(\alpha) - g'(\alpha) = 0$.

So if an agent with $\alpha^*$ has net benefit 0 then

$$q(\alpha) = \alpha u(x(\alpha)) - \int_{\alpha^*}^{\alpha} u(x(\omega))\, d\omega \,.$$

giving

$$\int_{\alpha^*}^{1} q(\alpha)\, d\alpha = \int_{\alpha^*}^{1} (2\alpha - 1) u(x(\alpha))\, d\alpha \,.$$

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So we seek to maximize a Lagrangian

$$L = \int_{\alpha^*}^{1} \Big[(\alpha + \lambda(2\alpha - 1))u(x(\alpha)) - (1 + \lambda)\alpha x(\alpha)\Big]\, d\alpha\,,$$

For $u(x) = \sqrt{x}$, this gives

$$x(\omega) = \left( \frac{2\lambda + 1}{2(\lambda + 1)} - \frac{\lambda}{2(\lambda + 1)\omega} \right)^2$$

We find the correct $\lambda$ by minimizing with respect to $\lambda$, giving $\lambda = 0.232206$. So for $\omega \geq 0.158566$,

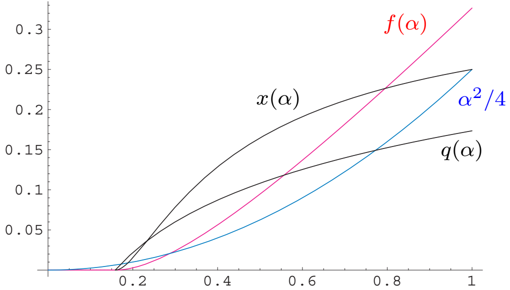
$$q(\omega) = 0.173521 + 0.0942239 \log \omega$$

$$x(\omega) = \left( 0.594224 - \frac{0.0942239}{\omega} \right)^2$$

and $q(\omega) = x(\omega) = 0$ for $\omega < 0.158566$ ($= \lambda/(1 + 2\lambda)$).

Note that agents with small $\alpha$ (less than $\alpha^* = 0.158566$) are prevented from participating.

# The optimal solution for $u(x) = \sqrt{x}$




The black lines show $q(\alpha)$ and $x(\alpha)$, with $q(\alpha) < x(\alpha)$ when $\alpha > 0.2339$.
The red line is the net benefit $f(\alpha) = tx(\alpha) - q(\alpha)$.
The the blue line is $\alpha^2/4$, the net benefit obtained acting alone.

Note that some agents would prefer self-provisioning.

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- Simple-minded sharing policies (like proportional sharing) may not to produce sufficient incentives for participants to contribute resources.
- Many new interesting problems!!!