Research areas:

Applications of mathematics to complex systems in manufacturing, communications, transport ....

Typical problem: Suppose n users share files with one another using a peer-to-peer network . If a total quantity of files Q are shared, the benefit to user i is $\theta_i \mathrm{u}(\mathrm{Q})$. We wish to choose Q to maximize

$$\Sigma_i \theta_i \mathrm{u}(\mathrm{Q}) - \mathrm{c}(\mathrm{Q})$$

and ask each user to pay a part of the cost $\mathrm{c}(\mathrm{Q})$.

What part of the cost can we ask of each user?

How can we cope with the problem of free-loading?

A Mathematician Plays Who Wants to be a Millionaire?

Professor Richard Weber
Statistical Laboratory
University of Cambridge

Who Wants to be a Millionaire?

15 questions away from a £1,000,000

1000000
500000 → 32000
250000 → 32000
125000 → 32000
64000 → 32000
32000 → 32000
16000 → 1000
8000 → 1000
4000 → 1000
2000 → 1000
1000 → 1000
500 → 0
300 → 0
200 → 0
100 → 0
0 → 0

Right answer ↗
Wrong answer →

Useful Mathematics

- Probability
- Optimization using Stochastic Dynamic Programming
- Statistics

Probability

The ideas of probability are fairly simple.

Example.

I will let you play the following game if you pay me £10.

You toss a fair coin.

If it comes up heads I will pay you £18.

If it comes up tails I will pay you £4.

Are you willing to pay £10 to play this game?

Yes. Because with probability 1/2 the coin shows heads and you win £18. With probability 1/2 it shows tails and you win £4. So on average you receive

$$(1/2)(£18)+(1/2)(£4) = £11$$

and £11 is more than £10.

Who Wants to be a Millionaire? Problem 1:

Suppose you have £500,000 and you know the right answer is either A or B (and you are sure it is definitely not C or D).

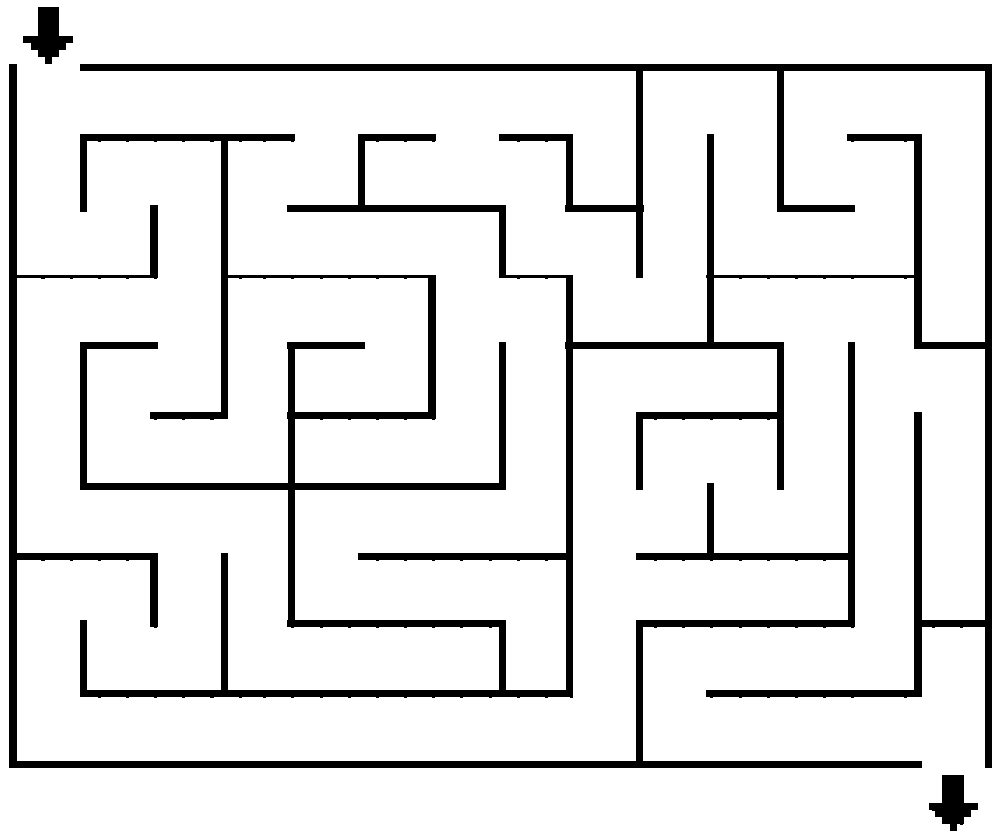
Should you guess?

This is a question about what to do near the end of the game (when we have already won £500,000).

Later we will think about what to do at the start of the game.

It is often a good problem-solving technique to work backwards.

This maze is more easily solved by working backwards.



"Life can only be understood backwards; but it must be lived forwards."

Soren Kierkegaard

Who Wants to be a Millionaire? Problem 1:

Suppose you have £500,000 and you know the right answer is either A or B (and you are sure it is definitely not C or D).

Should you guess?

If your don't play further you take home £500,000.

If you guess, then you either you are wrong and take home £32,000 or you are right and you take home £1,000,000.

These are equally likely. So your expected reward is

(1/2)(32000)+(1/2)(1000000) = £516,000.

This is what you could expect to get "on average".

Who Wants to be a Millionaire? Problem 2:

Suppose you have £500,000, you have no idea about the answer, but you do have the 50-50 lifeline.

What should you do?

If your don't play further you take home £500,000.

If you use the lifeline and then guess between the two remaining answers you take home £32,000 or £1,000,000.

These are equally likely. As before, your expected reward is

(1/2)(32000)+(1/2)(1000000) = £516,000.

Who Wants to be a Millionaire? Problem 3:

Suppose you have £250,000, no lifelines, and you think the probability that *A* is right is 0.5, that B is right is 0.3, that *C* is right is 0.2 and that D is definitely wrong (probability 0).

What should you do?

If your don't play further you take home £250,000.

If you *guess* *A* then with probability 0.5 you are wrong and take home only £32,000.

But with probability 0.5 you are right and now have £500,000.

So your expected reward is *at least**

0.5 (32000) + 0.5 (500000) = £266,000.

**at least*, because you can do even better if, once you get to £500,000, you know the answer to the £1,000,000 question.

Who Wants to be a Millionaire? Problem 4:

Suppose you have £250,000, the 50-50 lifeline is left, and you think that the probability that A is right is 0.5, that B is right is 0.3, that C is right is 0.2 and that D is definitely wrong. What should you do?

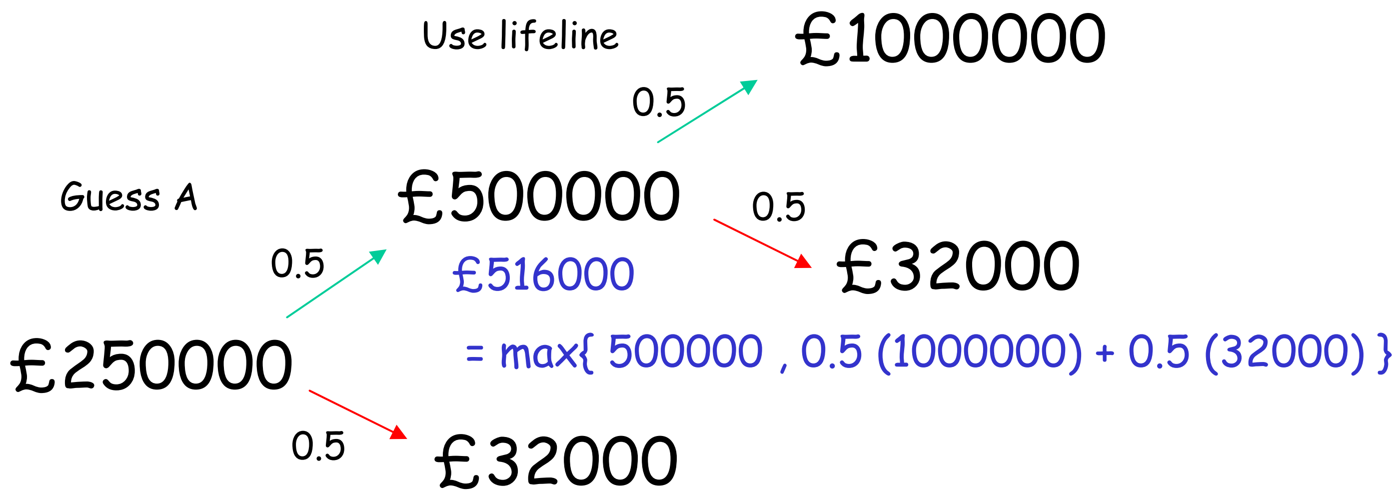
If your don't play further you take home £250,000.

If you guess A then with probability 0.5 you are wrong and take home £32,000. But with probability 0.5 you are right and reach £500,000, and you still have the 50-50 lifeline left.

In Problem 2 we saw that the expected reward from this point was at least £516,000.

So now our expected reward is at least

(0.5)(32000)+(0.5)(516000) = £274,000.


Use lifeline
£1000000
0.5
Guess A
£500000
0.5
£32000
0.5
£516000
£250000
= max{ 500000 , 0.5 (1000000) + 0.5 (32000) }
0.5
£32000

Guess A

£516000

0.5

£250000

£274000

0.5

£32000

= max{ 250000 , 0.5 (516000) + 0.5 (32000) }

Use lifeline

£1000000

0.5

£516000

Guess A

0.5

£32000

0.5

= max{ 500000 , 0.5 (1000000) + 0.5 (32000) }

£274000

0.5

£32000

= max{ 250000 , 0.5 (516000) + 0.5 (32000) }

Conclusion:
to be at £250,000,
believing A is right with probability 0.5,
and having the 50-50 lifeline left,
is worth

£274000

The approach we have used is called

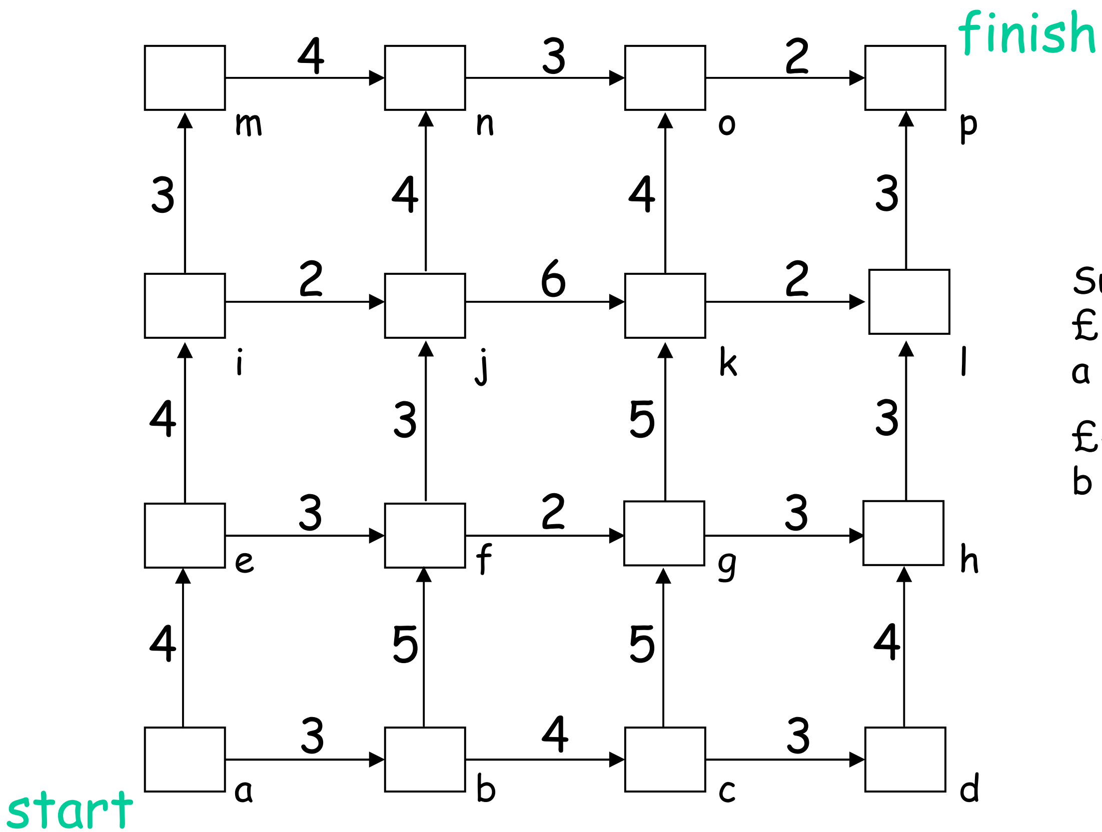
stochastic dynamic programming.

"Banking" strategy on *The Weakest Link*

Strategy	Optimal for
BANK at 20	$0.000 \le p \le 0.602$
BANK at 200	$0.602 < p \le 0.724$
BANK at 450	$0.724 < p \le 0.795$
BANK at 800	$0.795 < p \le 0.843$
BANK at 1000	$0.843 < p \le 1.000$

Optimization using Dynamic Programming

Problem 5: What is the longest path from a to p?

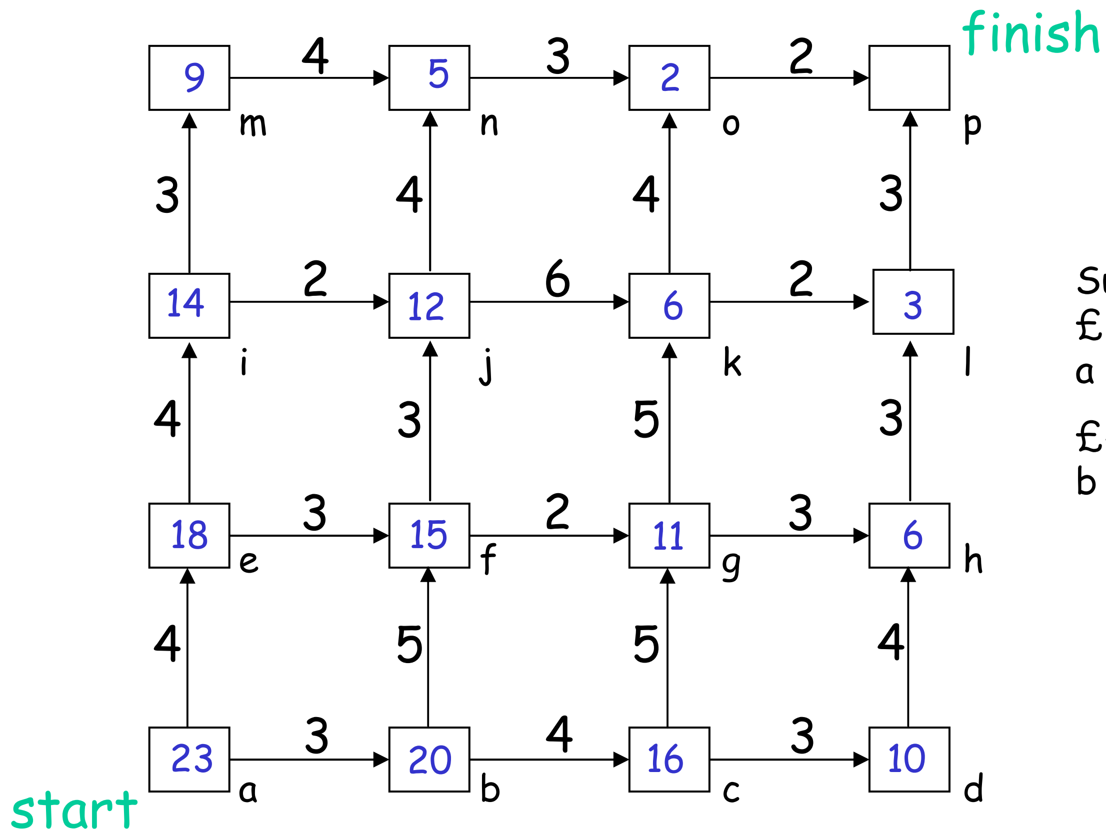


Suppose we collect £3 if we travel from a to b,

£4 if we travel from b to c, etc.

Note that there are 20 different paths from a to p.

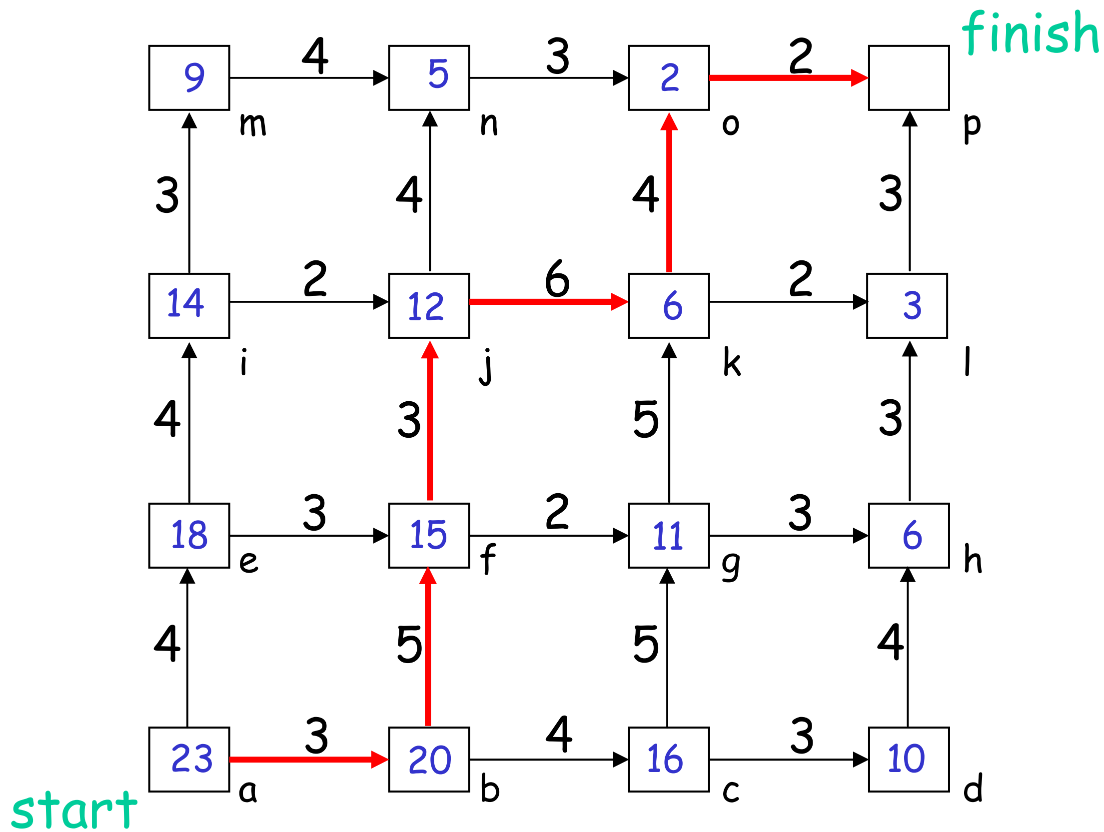
Problem 5: What is the longest path from a to p?



Suppose we collect £3 if we travel from a to b,

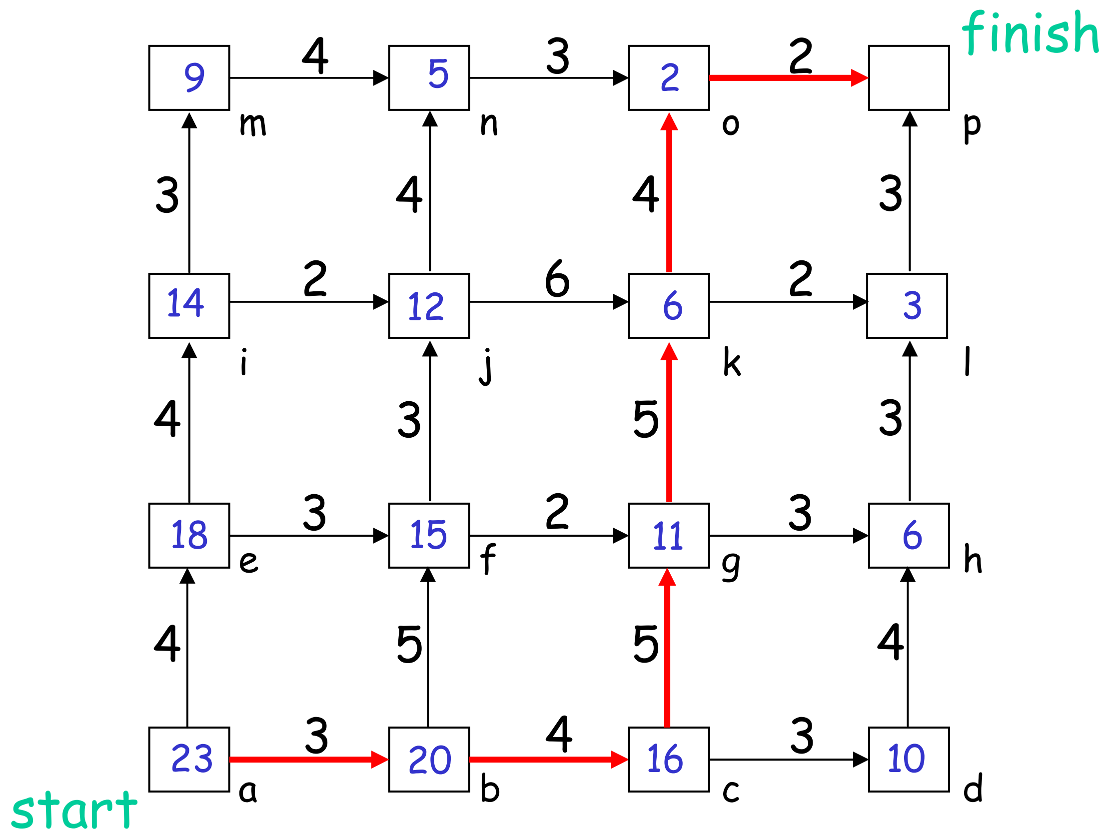
£4 if we travel from b to c, etc.

Problem 5: What is the longest path from a to p?



Problem 5: What is the longest path from a to p?

or



We have saved lots of work!

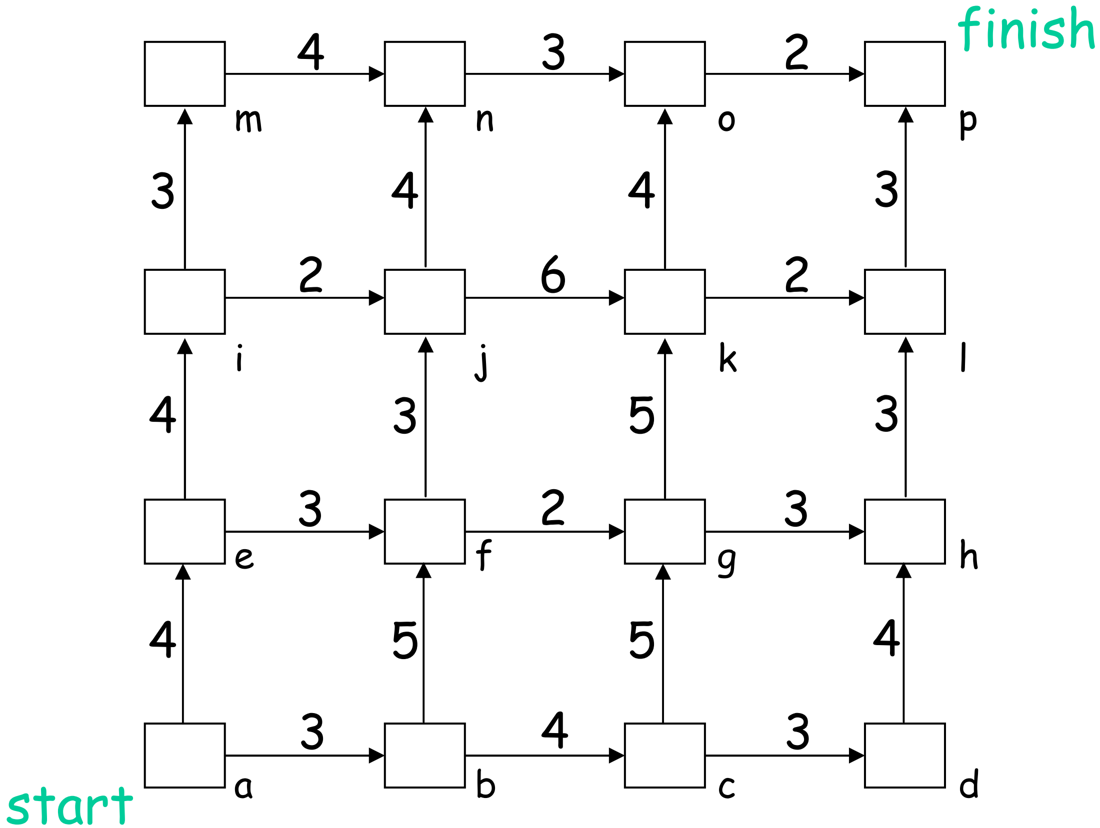
We might have calculated the length of each of the 20 possible paths.

Each calculation would have meant adding up 6 numbers.

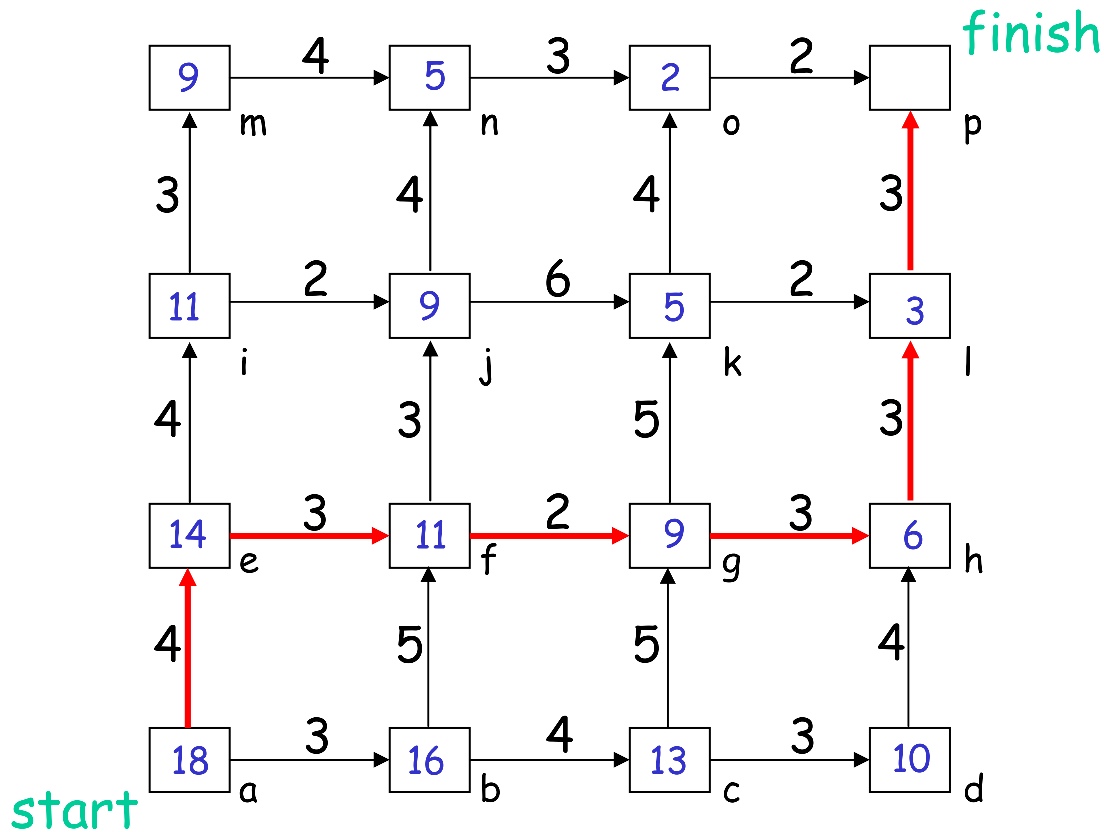
Having done that, we would then have to compare the 20 results to find the greatest one.

The calculation we have made using dynamic programming has required much less arithmetic.

Problem 6: What is the shortest path from a to p?

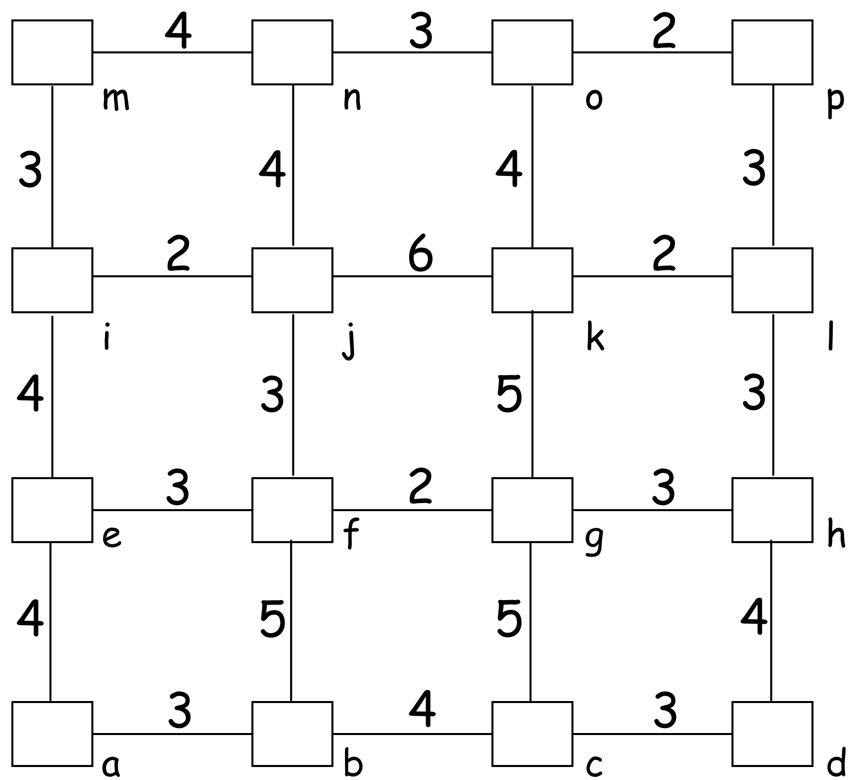


Problem 6: What is the shortest path from a to p?

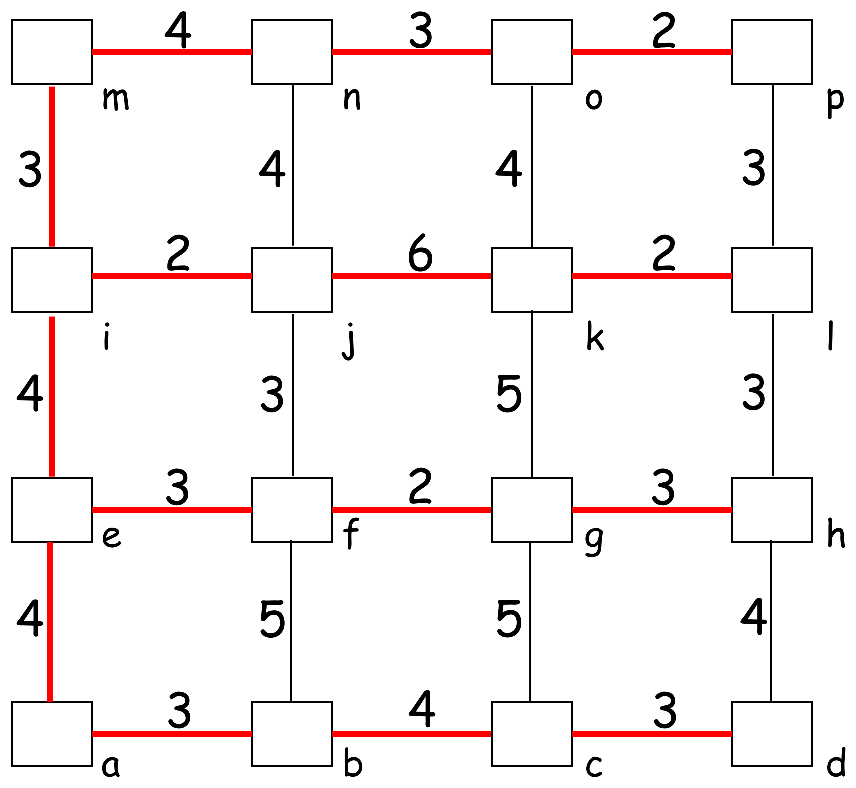


Cost = 18

Problem 7: What is the least length of wire needed to connect all the cities?



Problem 7: What is the least length of wire needed to connect all the cities?



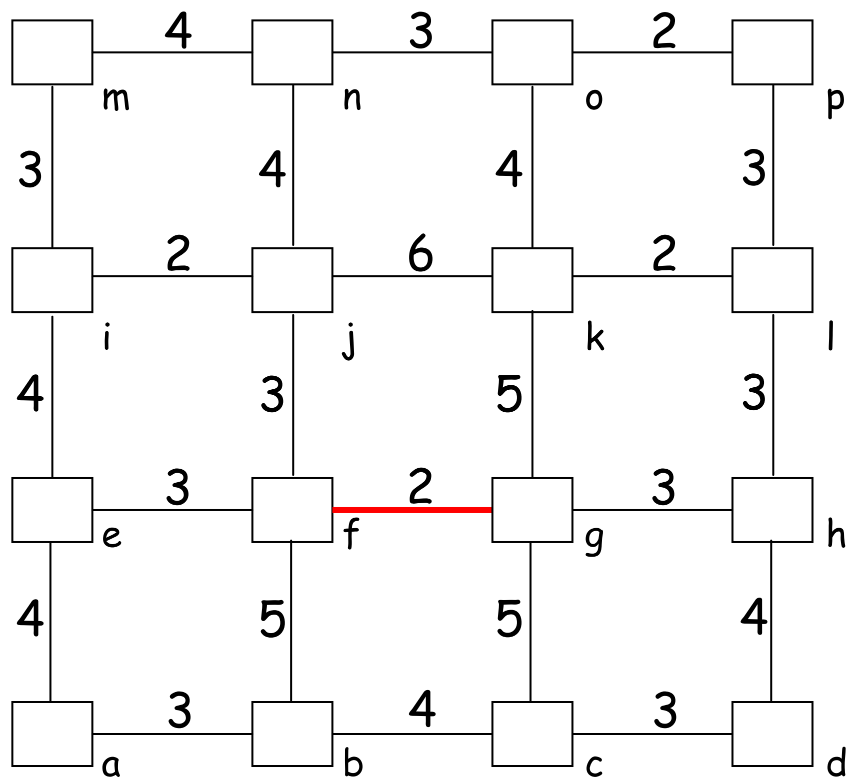
Cost = 48

An optimal algorithm for this problem is

Prim's algorithm

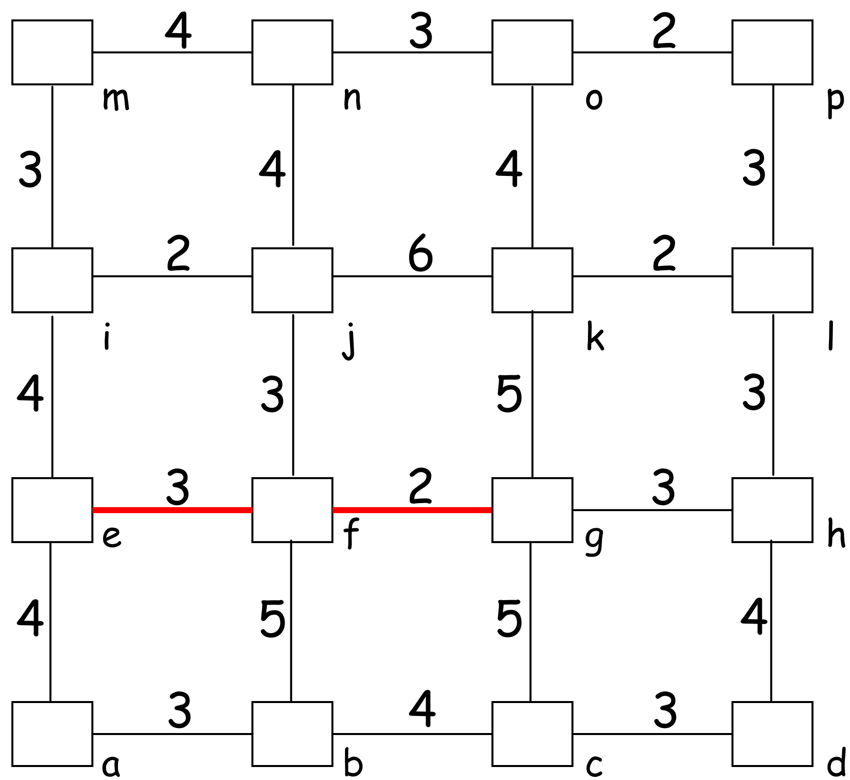
1. Let S be a set of cities. Initially, place any one city in the set S.
2. Find the city closest to those already in S and then add that city to S.
3. Repeat Step 2 until all cities are in S.

Problem 7: What is the least length of wire needed to connect all the cities?



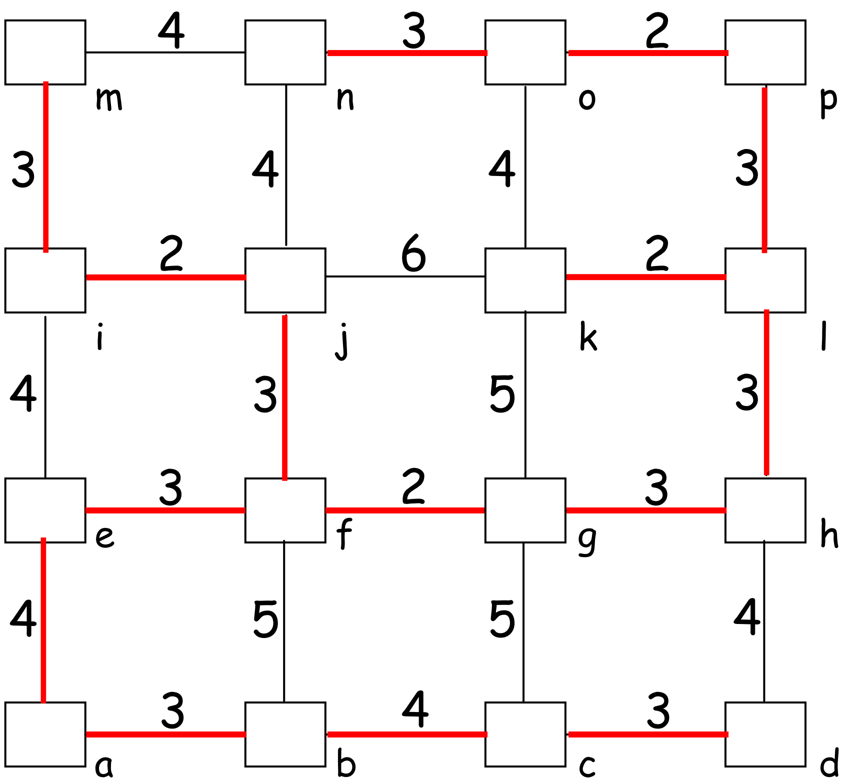
Suppose we start with f. We first add g.

Problem 7: What is the least length of wire needed to connect all the cities?



Now add e.

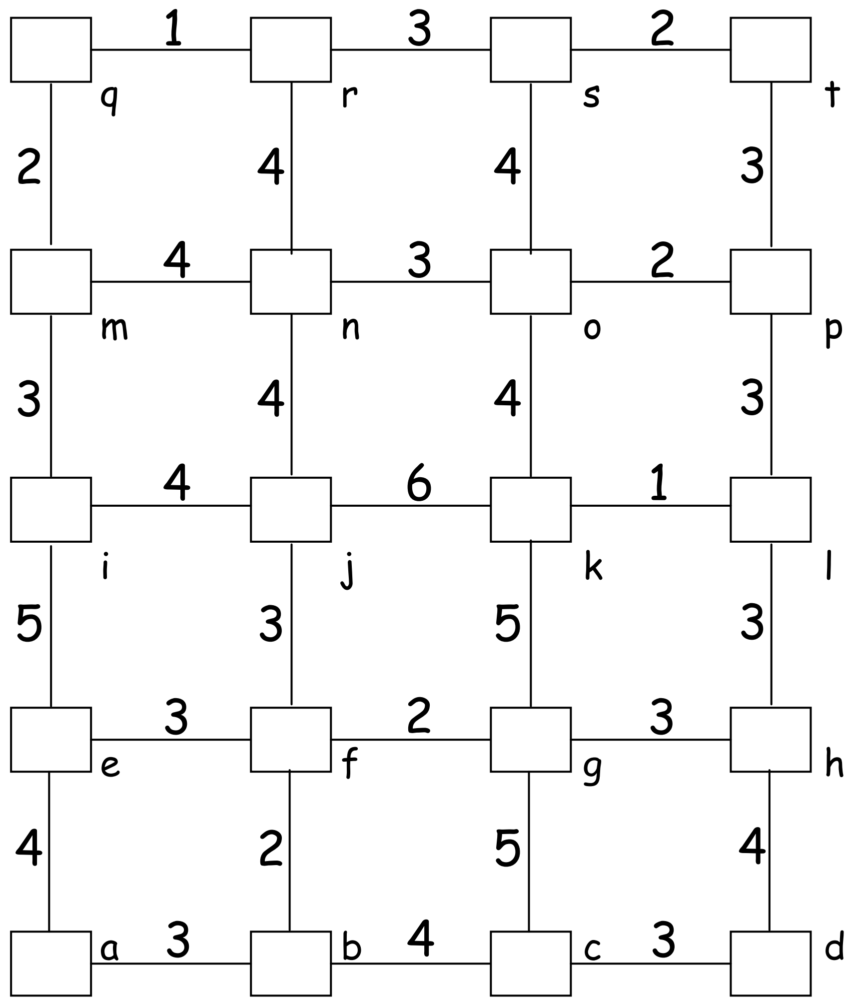
Problem 7: What is the least length of wire needed to connect all the cities?



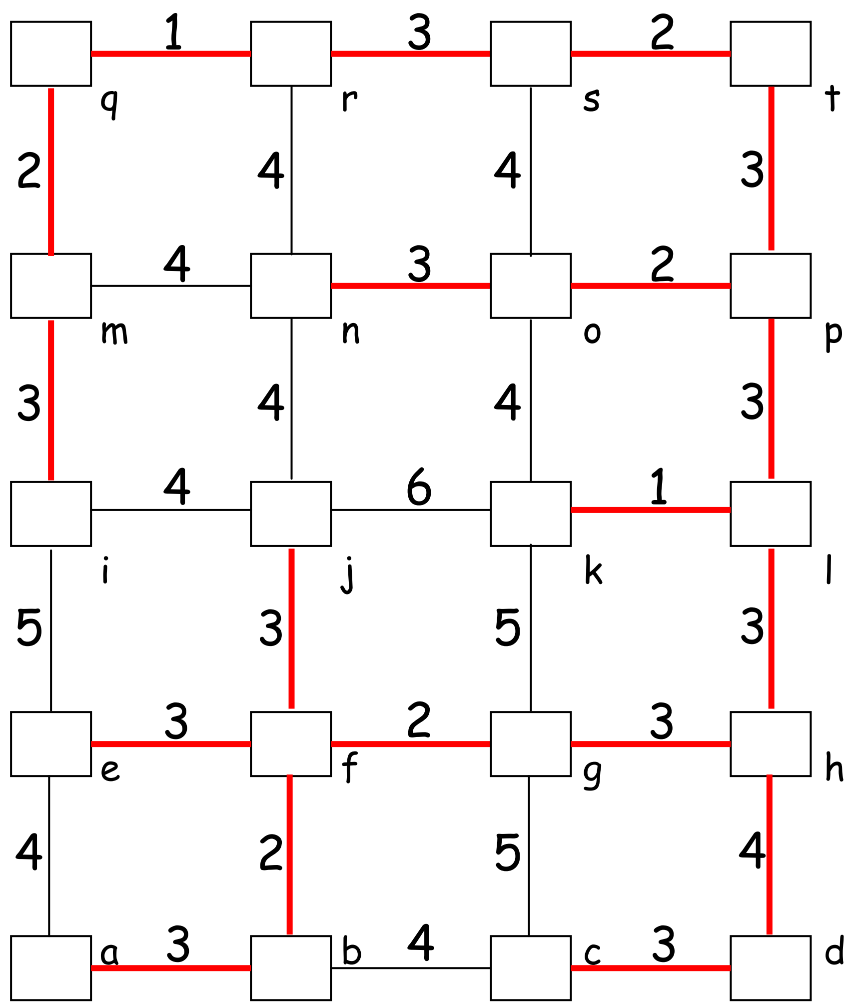
Total length = 43

The solution is a "minimum length spanning tree"

Problem 8: Apply Prim's algorithm to find the minimum spanning tree for the data below.

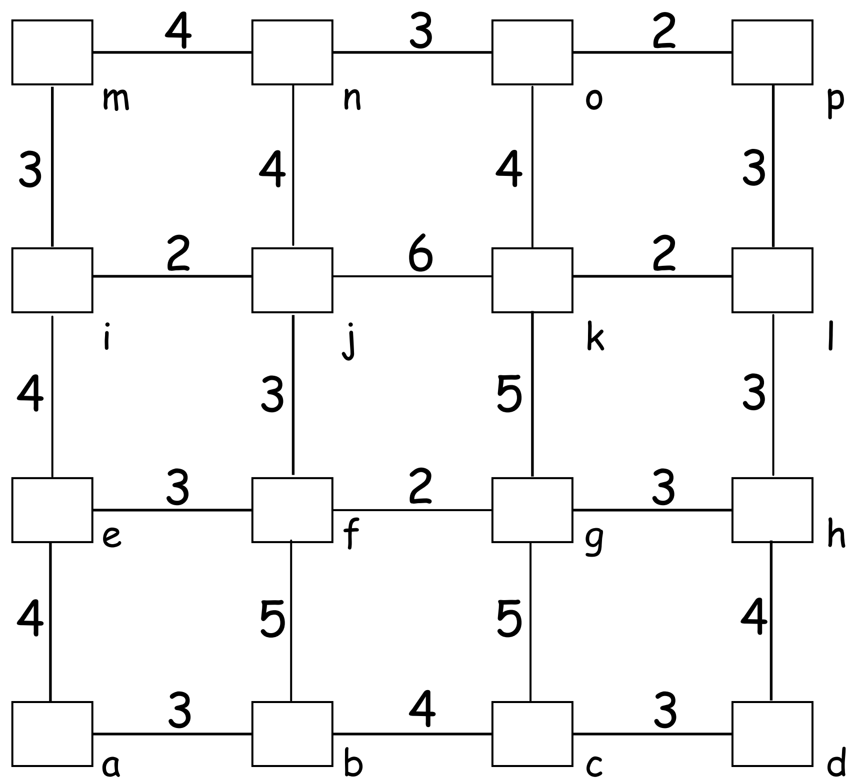


Problem 8: Try applying Prim's algorithm to find the minimum spanning tree for the data below.

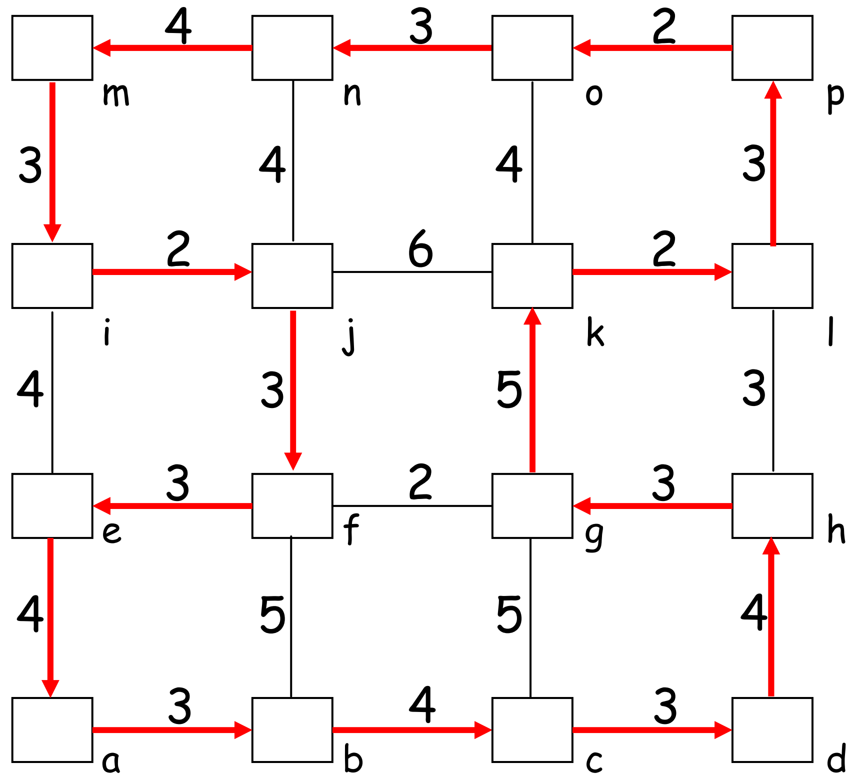


Cost = 49

Problem 9: What is the length of the shortest tour which starts at a, visits each city once and returns to a?



Problem 8: What is the length of the shortest tour which starts at a, visits each city once and returns to a?



Cost = 51.

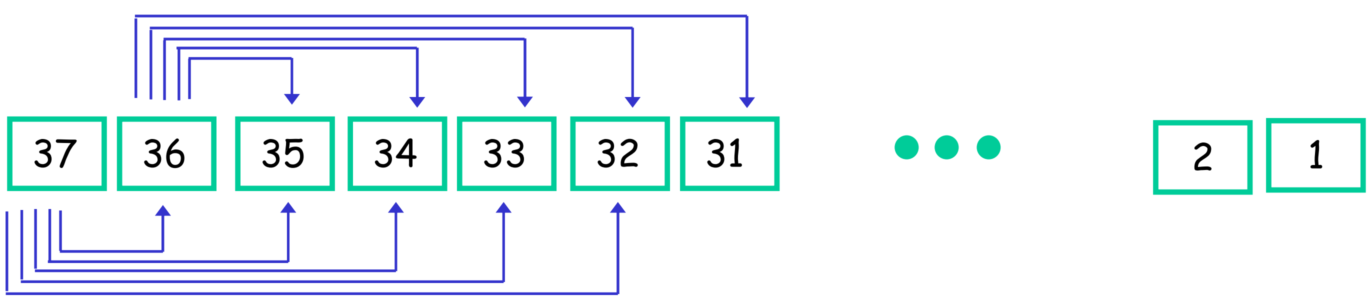
This is known as the "travelling salesman problem".

Prim's algorithm solves the minimum spanning tree problem efficiently. The work required to solve a problem with 20 cities is only about 2 times as hard as a problem with 10 cities.

However, there is no known method (like Prim's algorithm) for solving the travelling salesman problem efficiently. The work required to solve a problem with 20 cities is about 1000 times as hard as a problem with 10 cities.

Dynamic Programming is also useful for analyzing the game of Nim.

Rules for Nim: From a single heap of 37 matchsticks each of two players alternately subtracts any number from 1 to 5, except that the immediately previous deduction may not be repeated. A player wins when he leaves his opponent in a position in which he cannot take any matchsticks.



last	matches left																		
move	37	36	35	34	33	32	31	30	29	28	27	26	25	24	23	22	21	20	19
1	W	W	W	L	L	W	W	W	W	W	L	L	W	W	W	W	L	L	W
2	W	W	W	W	L	W	W	W	W	W	W	L	W	W	W	W	W	L	W
3	W	W	W	W	L	L	W	W	L	W	W	L	W	W	W	W	W	L	L
4	L	W	W	W	L	W	W	W	W	W	W	L	W	L	W	W	W	L	W
5	W	W	W	W	L	W	L	W	W	W	W	L	L	W	W	W	W	L	W

last	matches left																		
move	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0
1	W	W	W	W	L	L	W	W	W	W	W	L	W	W	W	W	W	L	L
2	W	W	W	W	W	L	W	W	W	W	W	L	W	W	W	W	W	W	L
3	W	W	L	W	W	L	W	W	W	W	W	L	L	W	W	L	W	W	L
4	W	W	W	W	W	L	W	L	W	W	W	L	W	W	L	W	W	W	L
5	L	W	W	W	W	L	L	W	W	W	W	L	W	L	W	W	W	W	L

The first player can win as long as the starting number of matchsticks is not 33, 26, 20, 13 or 7.

Another Nim: *From two heaps of 10 and 9 matchsticks, each of two players alternately subtracts any number from 1 to 3 (from one of the heaps). The player who takes the last matchstick wins.*

Try to complete the empty cells with W or L.

Left in pile 2

Left in pile 1

	10	9	8	7	6	5	4	3	2	1	0
10											
9											
8											
7											
6											
5											
4											
3											
2											
1										L	W
0											L

Is this a W or L?

Another Nim: From two heaps of 10 and 9 matchsticks, each of two players alternately subtracts any number from 1 to 3 (from one of the heaps). The player who takes the last matchstick wins.

Left in pile 2

Left in pile 1

	10	9	8	7	6	5	4	3	2	1	0
10	L	W	W	W	L	W	W	W	L	W	W
9		L	W	W	W	L	W	W	W	L	W
8			L	W	W	W	L	W	W	W	L
7				L	W	W	W	L	W	W	W
6					L	W	W	W	L	W	W
5						L	W	W	W	L	W
4							L	W	W	W	L
3								L	W	W	W
2									L	W	W
1										L	W
0											L

It is a win for the first player.

So how much is it worth to play Millionaire?

Question for £	100	200	300	500	1000	2000	4000	8000	16000	32000	64000	125000	250000	500000	1000000
Probability	1	1	1	1	1	0.99	0.8	0.7	0.5	0.4	0.35	0.3	0.3	0.29	0.28
answer	5679	5679	5679	5679	5679	5679	5726	6908	9440	17880	43200	59900	97400	167720	303040
don't answer	0	100	200	300	500	1000	2000	4000	8000	16000	32000	64000	125000	250000	500000
best	5679	5679	5679	5679	5679	5679	5726	6908	9440	17880	43200	64000	125000	250000	500000

The above table is for a game in which we have no lifelines. The probabilities with which we think we are going to be able to answer the questions are in blue.

At each point we must decide whether to quit or guess.

With these probabilities we should stop guessing once we reach £125,000.

So how much is it worth to play Millionaire with lifelines?

Now consider a game in which we have 3 lifelines.

At each point we must decide whether to quit, or guess using a lifeline, or guess not using a lifeline.

Let us suppose that using a lifeline we can guess correctly with probability 0.5.

As before, the probabilities with which we think we think we can answer the questions without using a lifeline are in blue.

So how much is Millionaire worth with lifelines?

Question for £	100	200	300	500	1000	2000	4000	8000	16000	32000	64000	125000	250000	500000	1000000
Probability	1	1	1	1	1	0.99	0.8	0.7	0.5	0.4	0.35	0.3	0.3	0.29	0.28
Probability using 50-50	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
0 50-50 lifeline left															
answer	5679	5679	5679	5679	5679	5679	5726	6908	9440	17880	43200	59900	97400	167720	303040
don't answer	0	100	200	300	500	1000	2000	4000	8000	16000	32000	64000	125000	250000	500000
best of the above	5679	5679	5679	5679	5679	5679	5726	6908	9440	17880	43200	64000	125000	250000	500000
1 50-50 lifeline left															
answer without 50-50	6849	6849	6849	6849	6849	6849	6908	8385	11550	19910	48275	64700	102200	172360	303040
answer using 50-50	2840	2840	2840	2840	2840	3363	3954	5220	9440	22100	48000	78500	141000	266000	516000
don't-answer	0	100	200	300	500	1000	2000	4000	8000	16000	32000	64000	125000	250000	500000
best of the above	6849	6849	6849	6849	6849	6849	6908	8385	11550	22100	48275	78500	141000	266000	516000
2 50-50 lifeline left															
answer without 50-50	7552	7552	7552	7552	7552	7552	7619	9273	12819	22700	51075	67100	104600	172360	303040
answer using 50-50	3424	3424	3424	3424	3424	3954	4693	6275	11550	24638	55250	86500	149000	274000	516000
don't-answer	0	100	200	300	500	1000	2000	4000	8000	16000	32000	64000	125000	250000	500000
best of the above	7552	7552	7552	7552	7552	7552	7619	9273	12819	24638	55250	86500	149000	274000	516000
3 50-50 lifeline left															
answer without 50-50	8519	8519	8519	8519	8519	8519	8595	10494	14563	24300	52475	68300	104600	172360	107891
answer using 50-50	3776	3776	3776	3776	3776	4309	5137	6909	12819	28125	59250	90500	153000	274000	516000
don't-answer	0	100	200	300	500	1000	2000	4000	8000	16000	32000	64000	125000	250000	500000
best of the above	8519	8519	8519	8519	8519	8519	8595	10494	14563	28125	59250	90500	153000	274000	516000

With these assumptions about probabilities, the value of the game to the player (right at the start) is £8,519.

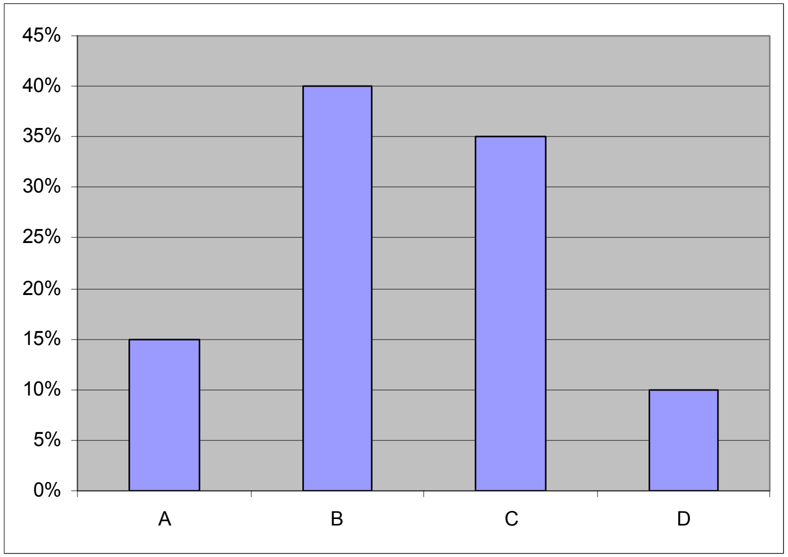
Statistics


In the Football World Cup 2002, Ireland lost in the second round on penalties to which team?
A: Sweden
B: Spain
C: Senegal
D: Saudi Arabia

A strategy for "Ask the audience"

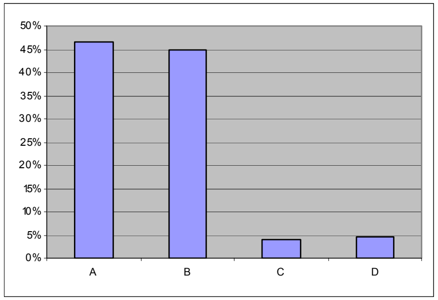
In Millionaire the contestant has a lifeline called "Ask the audience".

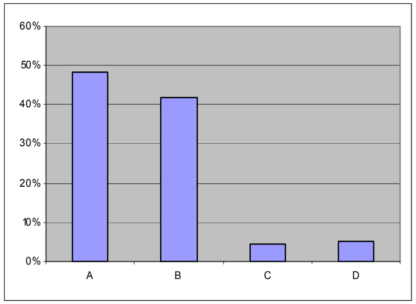
Typically, there are 200 people in audience; they are asked to vote, A, B, C, or D and results are:

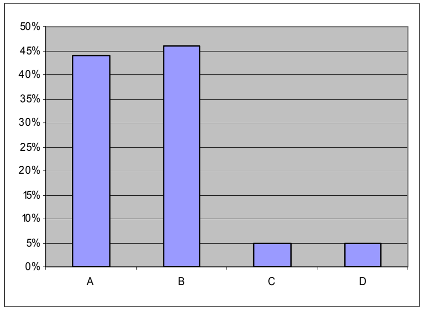


Suppose the answer is A.

20 people feel sure it is A.
15 people feel sure it is B.
125 people think it is probably A or B.
40 people have no idea whether it is A, B, C or D.



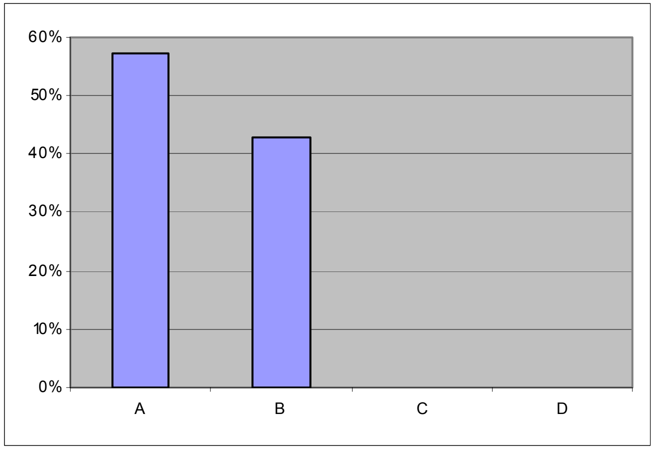




If we follow the majority, we can sometimes be right and sometimes wrong.

In fact, the majority will be for A about 70% of the time.

Suppose that only the people who feel they are sure vote. There are 20 votes for A and 15 for B. In this case we see



And so we make the right choice.


22% 58% 14% 6%
A B C D
In the Football World Cup 2002, Ireland lost in the second round on penalties to which team?
A: Sweden
B: Spain
C: Senegal
D: Saudi Arabia


In the Football World Cup 2002, Ireland lost in the second round on penalties to which team?
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B: Spain
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