The Bomber Problem

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Adams Society of St John's College, 30 November 2011

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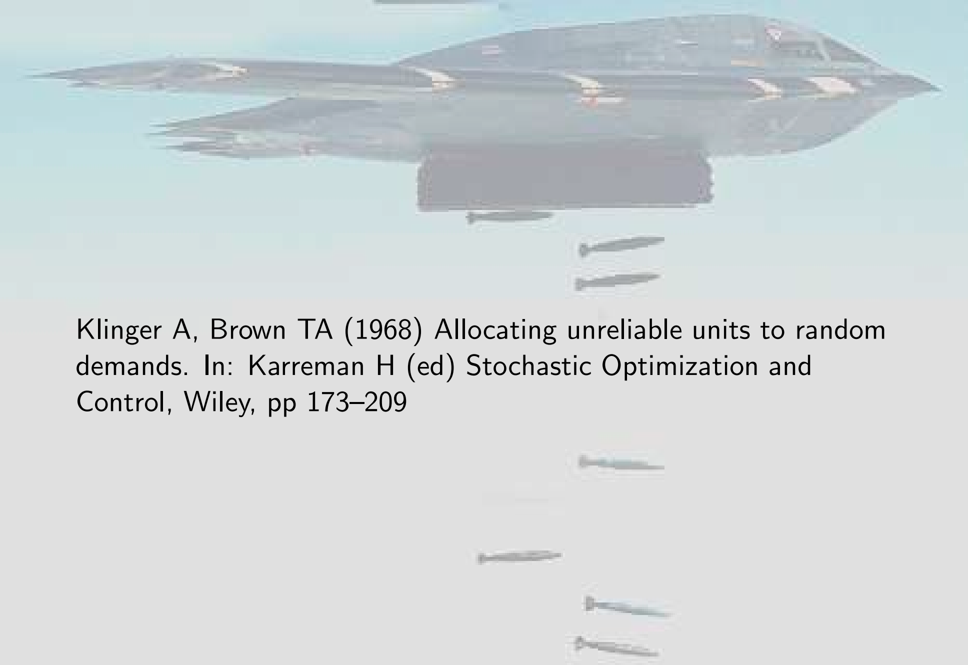

Klinger A, Brown TA (1968) Allocating unreliable units to random demands. In: Karreman H (ed) Stochastic Optimization and Control, Wiley, pp 173–209

Bomber Problem researchers

Yi-Ching Yao, Richard Weber, Larry Goldstein, Ester Samuel-Cahn, Jay Bartroff, Larry Shepp (Contributors to research on the Bomber Problem) at the 3rd International Workshop on Sequential Methods, Stanford June 16, 2011

A groundwater management problem

Burt (1965)

- x_t = level of water in an aquifer at start of day t.
- y_t = water extracted at start of day t.
- R_{t+1} = rainfall on day t.

$$x_{t+1} = x_t - y_t + R_{t+1}, \quad t = 0, 1, \ldots, T-1$$

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$$\underset{y_0,\ldots,y_{t-1}}{\text{maximize}}\, E\left[\sum_{s=0}^{T-1} a(y_t) - c(x_t, y_t)\right]$$

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- $c(x, y)$ is a cost, convex increasing in x,

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- $c(x, y)$ is a cost, convex increasing in x, perhaps

$$c(x, y) = \int_{x-y}^{x} \gamma(z)dz, \qquad \text{where } \gamma(z) \text{ is decreasing and convex.}$$

Stochastic dynamic programming

We can solve this problem by working backwards from time T.

$$F(x,s) = \max_{y\in[0,x]} \{a(y) - c(x,y) + \delta E F(x - y + R_t, s-1)\}$$
$$s = 1, \ldots, T$$
$$F(x,0) = 0.$$

$F(x,s)$ is the maximal reward that can be obtained over s remaining s $(= T - t - 1)$ days, starting with a water level of x.

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> *Life must be lived forward and understood backwards.*
> *(Kierkegaard)*

Sequential allocation problems

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Investment Derman, Lieberman and Ross (1975)

$$F(x,t) = q_t F(x, t-1) + p_t \max_{y \in [0,x]} \{a(y) + F(x-y, t-1)\}$$

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Fighter

$$F(n,t) = q_t F(n, t-1) + p_t \max_{k\in\{1,\dots,n\}}\{a(k) + F(n-k, t-1)\}$$

$F(n,0) = 0.$ k is remaining stock of missiles.

Fighter Problems

Fighter and Bomber Problems

Invincible Fighter Bartroff et al (2010)

$$F(n,t) = q_t F(n,t-1) + p_t \max_{k \in \{1,\ldots,n\}} \{a(k) + F(n-k,t-1)\}$$

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Frail Fighter Weber (1985)

$$F(n,t) = q_t F(n,t-1) + p_t \max_{k \in \{1,\ldots,n\}} \{a(k) + a(k)F(n-k,t-1)]\}$$

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General Fighter

$$F(n,t) = q_t F(n,t-1) + p_t \max_{k\in\{1,\dots,n\}} \{a(k) + c(k)F(n-k,t-1)\}$$

Might take $c(k) = a(k) + u(1-a(k))$.

Monotonicity properties **(A)**, **(B)** and **(C)**

$$F(n,t) = q_t F(n,t-1) + p_t \max_{k \in \{1,\ldots,n\}} \{a(k) + c(k) F(n-k, t-1)\}$$

Let $k(n,t)$ be the maximizing k in the above.

Intuitively obvious properties of an optimal policy are:

(A)	$k(n,t)$	$\searrow$	as $t \nearrow$
(B)	$k(n,t)$	$\nearrow$	as $n \nearrow$
(C)	$n - k(n,t)$	$\nearrow$	as $n \nearrow$

(A) (B) (C) s of fighter problems

If

(i) $\{p_t\}_{t=1,\ldots}$ is any sequence of probabilities;

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If

(i) $\{p_t\}_{t=1,\dots}$ is any sequence of probabilities;

(ii) $a(k)$ is nondecreasing and concave in k, then

(A) holds for the invincible fighter, in the strong sense that

$$\textbf{(A)}^* : \; k(n, p_{t-1}, \dots, p_1) \text{ is nonincreasing in each } p_i.$$

and for the frail fighter, nonincreasing in p_1.

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Does **(A)** *hold for the general fighter?*

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Does **(A)** *hold for the general fighter?*

(B) holds for the invincible fighter, but not frail fighter.

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Does **(A)** *hold for the general fighter?*

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If also,

(iii) $c(k)$ is nondecreasing and log-concave in k, then

(C) holds for the general fighter.

Bomber Problem

Klinger and Brown (1968)

With discrete ammunition, and attacks occurring as a Poisson process of rate 1, the continuous-time bomber problem (CBP) has defining equations:

$$\begin{aligned} P(n,t) &= P(\text{survive to until time } t) \\ &= e^{-t} + \int_0^t \max_{k\in\{1,\dots,n\}} c(k)P(n-k,s)e^{-(t-s)}\,ds. \\ P(n,0) &= 1. \end{aligned}$$

Bernoulli model: $c(k) = 1 - \theta^k$, a concave function of k.

Doubly-discrete Bomber Problem (DBP)

Aim is to survive t periods. With s periods to go, an attack occurs with probability p_s $(= 1 - q_s)$.

$$P(n,t) = q_t P(n,t-1) + p_t \max_{k \in \{1,\dots,n\}} c(k) P(n-k, t-1)$$

$$P(n,0) = 1.$$

Again we are interested in whether the following are true of false.

(A)	$k(n,t)$	$\searrow$	as $t \nearrow$	proved
(C)	$n - k(n,t)$	$\nearrow$	as $n \nearrow$	proved
(B)	$k(n,t)$	$\nearrow$	as $n \nearrow$	?

ALLEN KLINGER
THOMAS A. BROWN

Allocating Unreliable Units to Random Demands

1. Introduction. This paper is concerned with the allocation of a fixed inventory of unreliable units to a random number of demands. Qualitatively, only one unit of those allocated has to satisfy a demand. However, each unit will only satisfy a demand with a given probability; in that sense the units are "unreliable". The goal of an allocation strategy is to meet all demands encountered, that is, to have at least one allotted unit satisfying each demand. Two other measures of success will be discussed below: the expected number of consecutive demands met and the expected inventory remaining after meeting a sequence of demands successfully.

[1] Brown, T. A. and A. Klinger, "Calculating the Value of Bomber Defense Missiles", RM-5302 (Secret), The RAND Corporation, Santa Monica, 1967.

The denominator is obviously positive, so let us consider the numerator:

$$(1 + \int_0^{x+\Delta} f)(1 + \int_0^x g) - (1 + \int_0^x f)(1 + \int_0^{x+\Delta} g)$$

$$= \int_x^{x+\Delta} f\,[1 + \int_0^x g] - \int_x^{x+\Delta} g\,[1 + \int_0^x f]$$

$$\geq h(x) \int_x^{x+\Delta} g[1 + \int_0^x g] - \int_x^{x+\Delta} g[1 + \int_0^x f] > 0 .$$

It seems intuitively obvious that $\Psi(n, x) \geq \Psi(n-1, x)$; that is, with a larger supply one is always willing to make at least as generou an allocation. The extensive tables we computed have confirmed this conjecture. However, determined efforts by a number of people at RAND have failed to yield a rigorous proof that this is indeed the case If this could be proven, we could relax the hypothesis in our main the rem below.

(A) (B) (C)s and open problems

$$F(n,t) = q_t F(n,t-1) + p_t \max_{k\in\{1,\dots,n\}} \{a(k) + c(k)F(n-k,t-1)\}$$

$F(n,0)$	$a(k)$	$c(k)$	p_t	**(A)**	**(B)**	**(C)**	
0	Bernoulli	$= u + (1-u)a(k)$	$= p$	**?**	no	yes	general fighter
0	concave	$= u + (1-u)a(k)$	$= 1$	yes	no	yes	general fighter
0	concave	$= \delta$		(A)*	yes	yes	invincible fighter
0	concave	$= a(k)$		yes	no	yes	frail fighter

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1	0	Bernoulli	$= p$	yes	?	yes	bomber
1	0	log-concave	$= 1$	yes	yes	yes	bomber
1	0	log-concave		yes	no	yes	bomber
1	0	log-concave	$= p$	yes	no	yes	bomber
1	0	concave	$= p$	yes	no	yes	bomber

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1	0	log-concave		yes	no	yes	bomber
1	0	log-concave	$= p$	yes	no	yes	bomber
1	0	concave	$= p$	yes	no	yes	bomber
≥ 0	concave	log-concave				yes	bomber/fighter

(A) (B) (C) s of the bomber problem

Theorem 1 **(A)** *and* **(C)** *hold for the DBP (and CBP, CDBP and CCBP) under generous assumptions that*

(i) $\{p_t\}_{t=1,\dots}$ *is any sequence of probabilities (i.e. nonstationary).*

(iii) $c(k)$ *is any nondecreasing and log-concave function of* k.

Is **(B)** true under these same generous assumptions?

Proving **(C)**

Suppose that $k = k(n,t)$ and $k' = k(n+1,t)$

and **(C)** fails so that $n - k > n + 1 - k'$ (so $k' > k + 1$).

Let $k' \to k+1$ and $k \to k'-1$

Notice that

$$
\begin{aligned}
& c(k+1)P(n+1-(k+1)),t-1) + c(k'-1)P(n-(k'-1)),t-1) \\
& \quad - \Big[c(k')P(n+1-k',t-1) + c(k)P(n-k,t-1)\Big] \\
& = \Big[c(k+1) - c(k)\Big] P(n-k,t-1) \\
& \quad - \Big[c(k') - c(k'-1)\Big] P(n+1-k',t-1) \\
& > 0
\end{aligned}
$$

So this policy cannot have been optimal.

Suppose $c(k)$ is merely log-concave in k (rather than concave in k)

(A) and **(C)** are true.

(B) is not true.

$p_t = \frac{10}{11}$ for all t.

$\{c(0), c(1), c(2), c(3), c(4), \ldots\} = \{0, \frac{3}{16}, \frac{1}{2}, 1, 1, \ldots\}$.

Note that $c(i)^2 \geq c(i+1)c(i-1)$ for all $i \geq 1$.

$\{k(n,4)\}_{n=1,2,\ldots} = \{1, 1, 1, 2, 2, 3, 2, \ldots\}$.

i.e. $3 = k(6,4) > k(7,4) = 2$.

Suppose $c(k)$ is merely concave in k (rather than of Bernoulli form $c(k) = 1 - v\theta^k$)

(A) and **(C)** are true. **(B)** is not true.

Let ammunition be continuous (CDBP).

$$P(x,1) = q + pc(x)$$

$$P(x,2) = q^2 + qpc(x) + \max_y \Big\{ pqc(y) + p^2 c(y)c(x-y) \Big\}$$

We design $c(\cdot)$ so that it is not log-concave in the neighbourhood of $x = 3$, and so that in this neighbourhood,

$$y(x,2) = \arg\max_y \{c(y)P(x-y,1)\} = \frac{1+x}{2}.$$

$c(x)$ for which $P(x,2)$ is not log-concave in the neighbourhood of $x = 3$

$$c(x) = \min\left\{\tfrac{1}{96} + \tfrac{31}{96}x, \tfrac{17}{96} + \tfrac{31}{192}x, \tfrac{5}{12} + \tfrac{31}{384}x, 1\right\}$$

$$= \begin{cases} \frac{1}{96} + \frac{31}{96}x, & x \in \left[0, \frac{32}{31}\right] \\ \frac{17}{96} + \frac{31}{192}x, & x \in \left[\frac{32}{31}, \frac{92}{31}\right] \\ \frac{5}{12} + \frac{31}{384}x, & x \in \left[\frac{92}{31}, \frac{224}{31}\right] \\ 1, & x \geq \frac{224}{31} \end{cases}$$

(B) is not true under generous assumptions

$$y(32,3) = \arg \max_{y \in [0,5.24]} \Big[c(y) F(31.4 - y, 2) \Big] = 14.0079$$

$$y(33,3) = \arg \max_{y \in [0,5.25]} \Big[c(y) F(31.5 - y, 2) \Big] = 13.9174.$$

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(B) fails because

$$3.4027 = \arg \max_{y \in [0,6.39]} \Big[c(y) P(6.39 - y, 2) \Big]$$

$$3.3965 = \arg \max_{y \in [0,6.40]} \Big[c(y) P(6.40 - y, 2) \Big].$$

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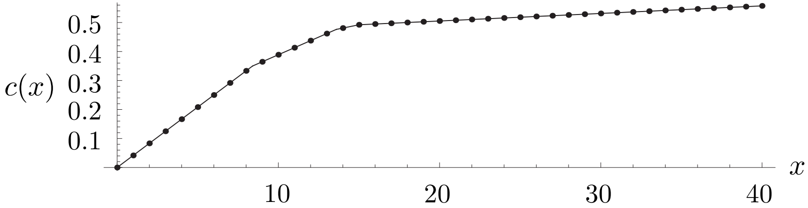
$6.39 - 3.4027 = 2.9873$
$6.40 - 3.3965 = 3.0035$
lie just either side of $x = 3$, where $P(x,2)$ is not log-concave.

Discrete ammunition counterexample

$$c(x) = \min\left\{\tfrac{1}{24}x, \tfrac{7}{48} + \tfrac{7}{288}x, \tfrac{371}{1152} + \tfrac{49}{4320}x, \tfrac{29}{64} + \tfrac{1}{384}x, 1\right\}.$$



$$\{k(n,2)\}_{n=1}^{40} = \{1,2,3,4,5,6,7,8,9,9,9,9,10,10,11,11,12,12,13,13,$$
$$13,14,14,14,14,14,14,15,15,15,16,17,18,19,20,21,22,23,24,25\}$$

$$\{k(n,3)\}_{n=1}^{40} = \{1,2,3,4,5,6,6,7,7,8,8,8,8,9,9,9,9,10,11,12,$$
$$13,13,13,13,14,14,14,14,14,14,15,14,15,14,14,15,15,15,15,15\}.$$

So $k(31,3) = 15 > 14 = k(32,3)$, in contradiction to **(B)**.

Log-concavity of $P(n, t)$

1. **(A)** follows from $\frac{P(n+1,t)}{P(n,t)} \nearrow$ in t.

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   $P(n,1) = q + pc(n)$ is concave in n.
   $P(n,2)$ is log-concave in n (for $c(k) = 1 - \theta^k$ model).
   $P(n,t)$ is not necessarily log-concave when $t \geq 3$.

Log-concavity of $P(n,t)$ can fail in DBP

$P(n,t)$ fails to be log-concave when

$$\Delta(n,t) = \frac{P(n+1,t)}{P(n,t)} - \frac{P(n,t)}{P(n-1,t)} > 0$$

for some n, t and some p, θ.

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$p = 0.58$, $\theta = 0.6$, $\Delta(8,3) = \frac{93682400617500}{668426731570135139} = 0.0001402$.

Simons and Yao (1990)

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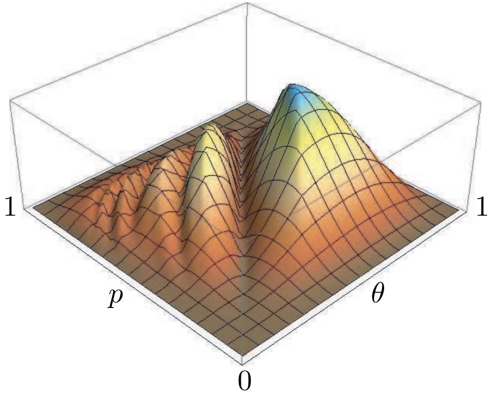
Simons and Yao (1990)

$p = 0.7207,\ \theta = 0.7254,\ \Delta(8,3) = 0.0004779.$

(most positive Δ found)

Regions of Log-concavity in DBP



$P(8,3)^2 - P(7,3)P(9,3)$ as a function of p and θ.
The region where this quantity is negative lies in the central trench, where p is a bit less than θ.

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1. I know of no example where $\Delta(n,t) > 0$ for $n < 8$.

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   E.g. $\theta = p = 99/100$, $\Delta(n, 3) > 0$ for $n = 16, 22, 28, 34, \ldots$.

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   E.g. $\theta = p = 99/100$, $\Delta(n,3) > 0$ for $n = 16, 22, 28, 34, \ldots$.
3. Log-concavity can fail for arbitrarily large t.
   Take p a bit less than θ and both approaching 1.

Log-concavity of $P(n,t)$ can fail in DBP

1. I know of no example where $\Delta(n,t) > 0$ for $n < 8$.
2. Log-concavity can fail for arbitrarily large n.

   E.g. $\theta = p = 99/100$, $\Delta(n,3) > 0$ for $n = 16, 22, 28, 34, \dots$.
3. Log-concavity can fail for arbitrarily large t.
   Take p a bit less than θ and both approaching 1.
4. Continuous time is the limit as $p \to 0$.
   So what about small p?

   For $p = 0.01$, $\theta = 0.01000048$, $\Delta(8,3) = 4.58768 \times 10^{-15}$.
   (This really is positive; checked in exact arithmetic).

Log-concavity in CBP

No examples have (yet!) been found in CBP for which $P(n, t)$ is not log-concave (continuous time, discrete ammunition).

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However, for a slightly different model $P(n,t)$ fails to be log-concave (and nonetheless **(B)** appears to hold).

We take $c(k) = 1 - (7/8)^k$ and make the restriction that only 1, 2 or 3 missiles may be fired.

Log-concavity in CBP

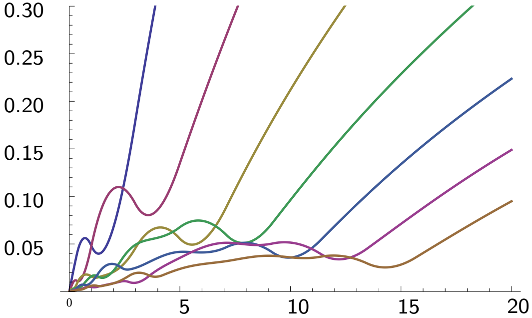

Figure: $-\Delta(n,t) = P(n,t)/P(n-1,t) - P(n+1,t)/P(n,t)$, for the continuous-time bomber problem with $\theta = 1/2$, for $0 \le t \le 20$ and $n = 2, \dots, 8$ (reading left to right across the asymptotes). Although we see that $P(n,t)$ is log-concave in n, the fact that these functions are not monotone increasing, in either n or t, means that it is probably difficult to prove that $P(n,t)$ is log-concave in n by some sort of induction on n, or using differential equations in t.

Continuous ammunition

CDBP and CCBP (continuous ammunition)

$$P(x,t) = qP(x,t-1) + p \max_{0<y\leq x} c(y)P(x-y,t-1)$$

or

$$\frac{d}{dt}P(x,t) = \max_{0<y\leq x} c(y)P(x-y,t)$$

1. $P(x,t)$ is log-concave in $x \iff$ **(B)** is true.

$$P(x,t)P''(x,t) - P'(x,t)^2 < 0.$$

Towards an iterative approach to proof of **(B)**

Consider iterating, from a start of $P_0(n,t)=1$, with

$$P_i(n,t) = e^{-t} + \int_0^t \max_{k\in\{1,\dots,n\}} c(k) P_{i-1}(n-k,s) e^{-(t-s)}\, ds$$

Might we inductively show that $P_i(n,t)$ is log-concave?
This poses a problem of maximizing the probability of surviving until time t, or until the first i attacks have been repelled.
Discrete time equivalent problem is

$$P_i(n,t) = q^t + \sum_{s=1}^{t-1} \max_{k\in\{1,\dots,n\}} c(k) P_{i-1}(n-k,s)\, q^{t-1-s} p$$

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Denote the maximizer of $c(k)P_i(n-k,s-1)$ as $k_i(n,s)$.

With $p = 1/2$, $c(k) = 1-(3/5)^k$, we find
$k_8(7,18) = 2 > k_8(8,18) = 1$.

So **(B)** does not hold for $k_8(n,18)$.

Also, rather surprisingly, $k_7(7,17) = 1$ and $k_8(7,17) = 2$.

Varying the final missile's miss probability **(B)**

Suppose that if the last missile is fired in a volley of k then

$$a(k) = 1 - \psi\theta^{k-1}, \quad v \in [\theta, 1].$$

Might we find $k(n, t, \psi)$ nonincreasing in ψ so that
$k(n, t) = k(n, t, \theta) \geq k(n, t, 1) = k(n - 1, t)$?

No counterexample to this has (yet) been found.

Another variation in which **(B)** fails

Suppose the boundary condition $P(n,0)=1$ is changed to

$$P(0,0) = 1$$
$$P(n,0) = 1+\lambda, \quad n = 1,2,\ldots$$

Then $k(n,t) \to k(n-1,t)$ as $\lambda \to \infty$.

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Then $k(n,t) \to k(n-1,t)$ as $\lambda \to \infty$.

But with $p = \theta = 3/5$, we find $p(8, 3, \lambda)$ is not nonincreasing in λ, and indeed

$k(8, 3, 0.6) = 4$ and $k(9, 3, 0.6) = 3$.

So **(B)** fails, with this slight change of boundary condition.

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$$\begin{aligned} P(0,0) &= 1 \\ P(n,0) &= 1 + \lambda, \quad n = 1, 2, \ldots \end{aligned}$$

Then $k(n,t) \to k(n-1,t)$ as $\lambda \to \infty$.

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$k(8,3,0.6) = 4$ and $k(9,3,0.6) = 3$.

So **(B)** fails, with this slight change of boundary condition.

Interestingly, for a boundary condition of $P(n,0) = n$, we find no counterexample to $P(n,t)$ being log-concave.

Special cases when **(B)** is true

1. DBP: $k(n+1,t) \geq k(n,t)$ for $t \leq 3$ or $n \leq 3$.

Special cases when **(B)** is true

1. DBP: $k(n+1,t) \geq k(n,t)$ for $t \leq 3$ or $n \leq 3$.
2. DBP: $k(n+1,t) = 1 \implies k(n,t) = 1$.

Conclusions

1. Proofs of **(A)** and **(C)** make no special use of $c(k) = 1 - \theta^k$.
   In discrete-time models they do not need $p_t = p$.
   They need only that $c(k)$ be log-concave.
   Yet **(B)** does not hold under such generous assumptions.

Conclusions

1. Proofs of **(A)** and **(C)** make no special use of $c(k) = 1 - \theta^k$.
   In discrete-time models they do not need $p_t = p$.
   They need only that $c(k)$ be log-concave.
   Yet **(B)** does not hold under such generous assumptions.

2. Experimental evidence still suggests the following are true (in the doubly discrete versions of the problems):

   (A) in the general fighter problem, when p_t is nonstationary and $a(k) = 1 - \theta^k$, $c(k) = 1 - v\theta^k$.

   (B) in the bomber problem, when p_t is nonstationary and $c(k) = 1 - \theta^k$.

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