Statistics Examples Sheet 3

This examples sheet covers material of the lectures 11–16 and is appropriate for your third supervision. A copy of this sheet can be found at: `http://www.statslab.cam.ac.uk/~rrw1/stats/`

On this examples sheet *numeric answers* are required and you will also need to consult statistical tables. You can use a pocket calculator. A computer spreadsheet is also a good way to do the calculations.

1. (Lecture 11, The t-test) A sample of nine pieces of steel wire of type A yields the following breaking strengths: 11.99, 12.02, 12.03, 12.09, 12.14, 12.16, 12.16, 12.16, 12.23. Use a t-test to test for any difference in mean breaking strength between wire A and a standard type whose mean breaking strength is known to be 12.10 units.

2. (Lecture 11, The t-test) The following figures give the % extension under a given load for two random samples of lengths of yarn, the first sample being taken before washing and the second after six washings.

Before washing:	12.3	13.7	10.4	11.4	14.9	12.6	
After 6 washings:	15.7	10.3	12.6	14.5	12.6	13.8	11.9

Do they provide any justification for concluding that extensibility is affected by washing? (You may assume that the standard deviation is unaltered by the washings.)

3. (Lecture 11, The t-test) In another experiment with the same type of yarn, six lengths of yarn were selected at random and each length cut into two halves. One of the halves was tested for extension without washing, and the other after six washings, giving the following % extensions:

Length:	A	B	C	D	E	F
Before washing:	13.9	12.5	11.0	11.8	10.8	14.6
After 6 washings:	14.7	12.1	13.2	13.6	11.5	15.4

What evidence do these provide as to the effect of washing on extensibility? Why should this experiment be analysed differently from the previous one?

4. (Lecture 11, The t-test) The following data is sampled from a $N(\mu, \sigma^2)$ distribution where both μ and σ^2 are unknown.

6.82 6.07 3.74 6.87 5.92

For this data, $\sum_i x_i = 29.42$ and $\sum_i x_i^2 = 179.588$.

(a) Find a 95% confidence interval for μ and show that its width is about 3.16.

(b) Suppose it becomes known that the true value of σ^2 is 1. Show that the 95% confidence interval for μ now has a width of 1.76.
Note that the width of the confidence interval has turned out to be narrower when the true value of σ^2 is known. Will this always happen?

(c) Consider the event that the 95% confidence interval for μ is narrower when σ^2 is known than it is when σ^2 is unknown, given that both intervals are computed from the same data, $X_1, \dots, X_n$, where these are IID samples from $N(\mu, \sigma^2)$.
Show that if $n = 5$ then this event has probability $1 - F_4(a)$, where $a = 4(1.96/2.78)^2 = 1.988$, and F_4 is the cdf of the χ_4^2 distribution. Verify from tables that this probability is a bit less than 0.75 (actually, 0.737).

5. (Lecture 12, The F-test and analysis of variance)

A butter-packing machine in a dairy packs butter in 250g packages though the actual weights packed vary. In a trial run, the following deviations from the nominal 250g were recorded:

$$5, 8, 0, 3, -1, 1, 6, 5, 8, 4, 9, 0, 4.$$

It is proposed to replace the machine with a new model which, it is claimed, has smaller variability in package weights. A trial run on the new machine produced the following deviations from the nominal 250g:

$$3, -2, 1, 4, 0, 2, 1, 5, 3, 2, 1, 4.$$

Test a null hypothesis that the variability of new machine is no better than the old.

Suppose the variance of the new machine is actually one-third that of the old, i.e. $\sigma^2_{\text{new}} = \frac{1}{3}\sigma^2_{\text{old}}$. On the basis of 13 trials of the old machine and 12 trials of the new machine a 5% level test is to be made of $H_0 : \sigma^2_{\text{new}} = \sigma^2_{\text{old}}$ against $H_1 : \sigma^2_{\text{new}} < \sigma^2_{\text{old}}$. What is the probability that a type II error will be made? *Hint:* $F^{(12,11)}_{0.55} = 0.932$.

6. (Lecture 12, The F-test and analysis of variance) An experiment was carried out to determine whether four specific temperatures used in the firing of bricks affected the density of the brick. The experiment led to the following data for 5 samples at 110° and 120°, and 4 at 130° and 140°:

$^\circ C$	110°	120°	130°	140°
data	20.8	20.6	20.9	20.8
	20.9	20.3	20.8	20.9
	20.7	20.2	20.8	20.9
	20.6	20.3	20.7	20.4
	20.7	20.5		

Carry out a one-way ANOVA to test for an effect of temperature upon density.

7. (Lecture 13, Linear regression and least squares) Show that the least squares estimator of β in a model $Y_i = \beta x_i + \epsilon_i$, (a regression through the origin), is $\hat{\beta} = \sum_i x_i Y_i / \sum_i x_i^2$. Under what assumptions is this also the MLE?

The range (in m) of a howitzer with muzzle velocity $V\ m/s$ fired at angle of elevation α is $g^{-1}V^2 \sin 2\alpha$, where $g = 9.81$. Suppose that 9 shells are fired at different angles of elevation, with the following results:

α	5	10	15	20	25	30	35	40	45
$\sin 2\alpha$	0.1736	0.3420	0.5	0.6428	0.7660	0.8660	0.9397	0.9848	1
range	4,860	9,580	14,080	18,100	21,550	24,350	26,400	27,700	28,300

(Ranges are given in m, and angles in degrees.) Estimate V from these data.

8. (Lecture 13, Linear regression and least squares) A beaker of liquid is heated, and then removed from the source of heat and left to cool in room whose temperature is 9°C. The temperature of the liquid is measured at the moment it is removed from the heat, and at intervals of 1 minute thereafter. The following results are observed:

time (secs)	0	1	2	3	4	5	6
temperature (°C)	100	82	60	50	40	32	28

Set up, and study, an appropriate regression model. *Hint: The temperature of the liquid decays towards room temperature exponentially with time.*

9. (Lecture 14, Hypothesis tests in regression models) Carry out a linear regression analysis on the following data:

x	−3	−3	−2	−1	0	1	1	2	2	3
y	114	110	111	107	107	108	104	105	103	101

Find a 95% confidence interval for β, the true slope of the regression line, and for the mean value of Y when $x = 1$.

10. (Lecture 14, Hypothesis tests in regression models) Five groups of 25 male fruit-flies have been kept under quite different conditions. Within each group longevity (y) is positively correlated with thorax size (x). Suppose that in the ith group

$$y_{ij} = \alpha_i + \beta_i x_{ij} + \epsilon_{ij}, \quad j = 1, \dots, 25,$$

where ϵ_{ij} are IID $N(0, \sigma^2)$. It is desired to test $H_0 : \beta_1 = \cdots = \beta_5$ against the alternative that H_0 is not true. On minimizing $S = \sum_{i=1}^{5} \sum_{j=1}^{25} (y_{ij} - \alpha_i - \beta_i x_{ij})^2$, with respect to $\alpha_1, \dots, \alpha_5$, $\beta_1, \dots, \beta_5$ under constraints appropriate to H_0 and H_1, we find minimized values of $R_0 = 12821.0$ and $R_1 = 12633.2$ respectively. Test H_0, saying how you determine the degrees of freedom used in your test. You may assume, by analogy with similar tests you have seen, appropriate distributional results for R_0, R_1 and $R_0 - R_1$. (The calculation required to answer this question is trivial. The important thing is to see if you can understand the rationale that underlies it.)

11. (Lecture 15, Computational methods) Twelve language students were asked to read an article in Japanese and then answer 25 multiple-choice questions about the article. Six students (randomly chosen) were presented with the questions written in English and six were presented with the questions in Japanese. Their scores were

Questions written in English:	12	17	19	22	25	25
Questions written in Japanese:	10	11	13	16	21	24

It is desired to test a null hypothesis that the language in which the questions are presented makes no difference to the students' performances. To do this a computer program was used to randomly sample 6 numbers without replacement from the set of 12 numbers above and to compute their total. This process was repeated 200 times. Totals of 85, 87, 88, 89, 90, ..., 120, 121, 122, 123 and 130 were observed to occur 1, 2, 1, 3, 5,..., 5, 8, 3, 1 and 1 times respectively.

Use this information to conduct a 5% level test of the null hypothesis, explaining the thinking that underlies your test. What do you think are the advantages and disadvantages of this test (which is known as a 'permutation test') compared to the two-sample t-test for equality of means?

12. (Lecture 16, Decision theory and Bayesian inference) A random variable X is observed. If it comes from class A its distribution is $N(0, 1)$, and if it comes from class B its distribution is $N(1, 25)$. Classes A and B are equally likely, so the prior is $\mathbb{P}(A) = \mathbb{P}(B) = \frac{1}{2}$. Under 0–1 loss, what is the Bayes rule for classification (i.e., what rule minimizes the probability of misclassification)? How should $X = 1.5$ be classified? What is the probability that when $X = 1.5$ a misclassification occurs?

R Weber, January 22, 2008