MATHEMATICAL TRIPOS: PART IB **(1995)**

Statistics – Example Sheet 1 **AY**

[*First sheet of 3*]

1. Let $X_1, \ldots, X_n$ be independent Poisson random variables with X_j having parameter $j\lambda$, where $\lambda > 0$ is an unknown parameter. Find a sufficient statistic for λ. What is its distribution?

2. Suppose that $X_1, \ldots, X_n$ are independent random variables, $X_j \sim N(\theta, \sigma_j^2)$, where θ is an unknown parameter, but the σ_j^2, $j = 1, \ldots, n$ are known. Find a sufficient statistic for θ. What is its distribution? Find the MLE of θ, and find a 95% confidence interval for θ.

3. Suppose that $n > 2$, and

$$f(x|\lambda) = \begin{cases} \lambda e^{-\lambda x} & (x \geq 0), \\ 0 & (x < 0). \end{cases}$$

Find a sufficient statistic $T_n = T_n(X_1, \ldots, X_n)$ for λ, and write down its density. Obtain the maximum-likelihood estimator for λ and show that it is biased, but that some multiple of it is not. Find the distribution of $U_n = 2\lambda \sum_{i=1}^n X_i$ and hence obtain a 95% confidence interval for λ.

4. A random variable is uniformly distributed over $(0, \theta)$. Show that the maximum of a random sample of n values of this variable is sufficient for θ and that this is also the MLE for θ. Show also that a $100\gamma\%$ confidence interval for θ is $(y_n, y_n/(1-\gamma)^{1/n})$, y_n being the maximum of the sample.

5. In a genetics experiment, a sample of n individuals was found to include a, b, c of the three possible genotypes GG, Gg, gg respectively. The population frequency of a gene of type G is $\theta/(\theta+1)$, where θ is unknown, and it is assumed that the individuals are unrelated and that two genes in a single individual are independent. Show that the MLE of θ is $(2a+b)/(b+2c)$.

6. Let $X_1, \ldots, X_n$ be a random sample from a distribution with the density function

$$f(x|\alpha, \lambda) = \frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)}, \quad x > 0.$$

Find a sufficient statistic for (α, λ). Discuss how to obtain the MLE of (α, λ).

7. The random variables $X_1, \ldots, X_n$ are IID Poisson with mean μ. Write down a simple unbiased estimator of $\theta = P(X_1 = 0)$. Is it a function of the sufficient statistic for μ? If not, find a better estimate by conditioning on the sufficient statistic.

8. Suppose that the number of defects on a roll of magnetic recording tape has a Poisson distribution for which the mean λ is known to be either 1 or 1.5. Suppose the prior mass function for λ is

$$p_\lambda(1) = 0.4, \quad p_\lambda(1.5) = 0.6\ .$$

If a collection of rolls of tape are found to have 3, 1, 4, 6, 2 defects respectively, what is the posterior distribution for λ?

9. Suppose $X_1, \ldots, X_n$ are IID from a distribution uniform on $(\theta - \frac{1}{2}, \theta + \frac{1}{2})$, and that the prior for θ is uniform on (10,20). Calculate the posterior distribution for θ, and write down point estimates for θ under quadratic and absolute error loss functions.

10. Suppose $X_1, \ldots, X_n$ form a random sample from the following pdf:

$$f(x|\theta) = \begin{cases} \theta x^{\theta-1} & 0 < x < 1\,, \\ 0, & \text{otherwise}\,, \end{cases}$$

and that the prior for θ, $p(\theta)$ is Gamma$\,(\alpha, \beta)$, $\alpha > 0, \beta > 0$. Calculate a point estimate for θ under quadratic loss function.

11. Suppose that X is distributed as a Binomial random variable with parameters n and θ. Calculate a point estimate, based on the single observation X, for θ when the prior distribution for θ is the uniform distribution on $[0, 1]$ and the loss function is

$$L(a, \theta) = \frac{(\theta - a)^2}{\theta(1-\theta)}\ .$$