Symmetric Rendezvous Search

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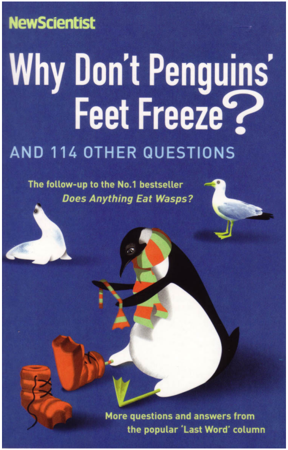
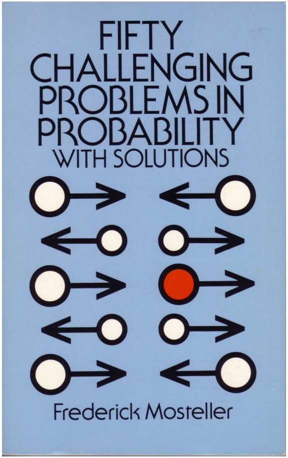
Aisle miles (2006)



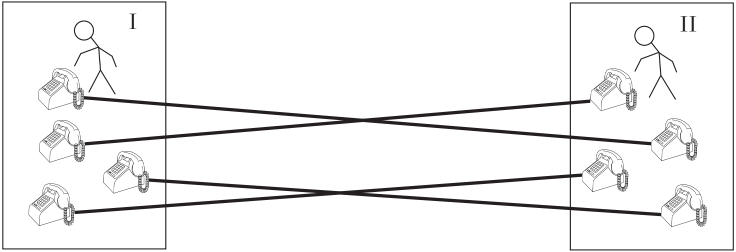
Two people lose each other while wandering through the aisles of a large supermarket. The height of the shelves precludes aisle-to-aisle visibility. One person wishes to find the other. Should that person stop moving and remain in a single visible site while the other person continues to move through the aisles? Or would an encounter or sighting occur sooner if both were moving through the aisles? (David Kafkewitz, in letter to the *New Scientist*)

Quo vadis? (Mosteller, 1965)



Two strangers who have a private recognition signal agree to meet on a certain Thursday at 12 noon in New York City, a town familiar to neither to discuss an important business deal, but later they discover that they have not chosen a meeting place, and neither can reach the other because both have embarked on trips. If they try nevertheless to meet, where should they go?

Telephone coordination game (Alpern, 1976)



In each of two rooms, there are n telephones randomly strewn about. The phones are connected pairwise in some unknown fashion. There is a player in each room. In each period $1, 2, \ldots$, each player picks up a phone and says "hello", until the first time that they hear one another. The common aim of the players is to minimize the expected number of periods required to meet.

Symmetric rendezvous search on K_n

Assumptions

1. There are n locations, which are connected as the complete graph, K_n.
2. Two players are randomly placed at two distinct locations.
3. The players have no common labelling of the locations.
4. At steps, $1, 2, \ldots$, each player relocates himself at one of the n locations.
5. Players adopt an identical (randomizing) strategy, in which the place that a player locates himself at step t can be only a function of where he has been at times $0, 1, \ldots, t-1$.

What should their common strategy be if they are to meet in the least expected number of steps?

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Some possible strategies

Let T be the number of steps in which the players meet.

Move-at-random If at each discrete step $1, 2, \ldots$ each player were to locate himself at a randomly chosen location, then the expected time to meet would be n. E.g.,

$$ET = 1 + \frac{n-1}{n} ET \quad \Longrightarrow \quad ET = n \,.$$

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$$ET = 1 + \frac{n-1}{n} ET \implies ET = n\,.$$

Wait-for-mommy Suppose the players could break symmetry (or had some prior agreement). Now it is best for one player to remain stationary while the other tours all other locations in random order. They will meet (on average) half way through the tour. So

$$ET = \tfrac{1}{n-1}\big(1 + 2 + \cdots + (n-1)\big) = \tfrac{1}{2} n\,.$$

Wait-for-mommy

Theorem 1 *In the asymmetric rendezvous search game on* K_n *the optimal strategy is wait-for-mommy. (Anderson-Weber, 1990)*

Proof. Let I_j denote the event that the players meet at their jth step, irrespective of any meeting previously.

$P(I_j) \leq \frac{1}{n-1}$.

$P(T \leq k) = P\left(\bigcup_{j=1}^{k} I_j\right) \leq \sum_{j=1}^{k} P(I_j) \leq \frac{k}{n-1}$.

$ET \geq \sum_{k=0}^{n-1} P(T > k) \geq \sum_{k=0}^{n-1}\left(1 - \frac{k}{n-1}\right) = \frac{1}{2}n$.

The Anderson-Weber strategy

Motivated by the optimality of *wait-for-mommy* in the asymmetric case, Anderson and Weber (1990) proposed the following strategy:

AW : If rendezvous has not occurred within the first $(n-1)j$ steps then in the next $n-1$ steps each player should either stay at his initial location or tour the other $n-1$ locations in random order, with probabilities p and $1-p$, respectively, where p is to be chosen optimally.

On K_2, **AW** with $p=\frac{1}{2}$ is the same as *move-at-random*.

Let $w=\inf\{ET\}$, where the infimum is taken over all possible strategies.

The Anderson-Weber strategy on K_n

Suppose each player stays at home with probability p, or tours the other $n-1$ locations with probability $1-p$. Then

$$\begin{aligned} ET \approx\ & p^2(n-1+ET) \\ & + 2p(1-p)\tfrac{1}{2}(n-1) \\ & + (1-p)^2\left(\int_0^{n-1}\tfrac{t}{n-1}e^{-\frac{t}{n-1}}\,dt + e^{-1}(n-1+ET)\right). \end{aligned}$$

As $n \to \infty$ the minimizing p tends to 0.24749 and rendezvous value is $w \sim 0.8289n$.

The Anderson-Weber strategy on K_2

Theorem 2 *On K_2,* **AW** *minimizes $P(T > k)$ for all k.*

Proof.

$P(I_k) \leq 1/2 \implies P(I_k^c) \geq 1/2.$

Indeed, given players have not met by step $k-1$, we still have $P(I_k^c \,|\, I_1^c \cap \cdots \cap I_{k-1}^c) \geq 1/2.$

Thus $P(T > k) = P(I_1^c \cap \cdots \cap I_k^c) \geq 1/2^k.$

Corollary. $w = 2$ on K_2.

Proof. $ET = \sum_{k=0}^{\infty} P(T > k) \geq \sum_{k=0}^{\infty} 2^{-k} = 2.$

The Anderson-Weber strategy on K_3

On K_3, **AW** specifies that in each block of two consecutive steps, each player should, independently of the other, either stay at his initial location or tour the other two locations in random order, doing these with respective probabilities $p = \frac{1}{3}$ and $1 - p = \frac{2}{3}$.

AW gives $ET = \frac{5}{2}$, whereas *move-at-random* gives $ET = 3$.

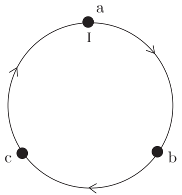
A new result is the following theorem (Weber, 2006).

Theorem 3 *On K_3,* **AW** *minimizes ET.*

A corollary of Theorem 2 is that $w = \frac{5}{2}$ on K_3.

Formulation of the problem

Suppose the three locations are arranged around a circle.



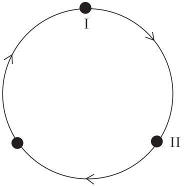
Each player calls his home location 'a'.

Each player chooses a direction he calls 'clockwise' and the labels that are one and two locations clockwise of home as 'b' and 'c' respectively.

A sequence of a player's moves can now be described.
E.g., a player's first 6 moves might be '$ababbc$'.

Suppose the problem is made a bit easier: the players are provided with the same notion of clockwise.
(We will see later that this does not actually help the players.)

Player II starts one position clockwise of Player I.



$$B_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

The matrix B_1 has '1' if after the first step they do not meet, and '0' if they do.

The rows of B_1 correspond to I playing a, b or c.

The columns of B_1 correspond to II playing a, b or c.

The minimum of $P(T > 1)$

If instead, Player II starts two steps clockwise of Player I, then the indicator matrix (for not meeting after one step) is $B_1^\top$. Let a bar over a matrix denote the mean of the matrix and its transpose. So

$$\bar{B}_1 = \tfrac{1}{2}(B_1 + B_1^\top)\,.$$

Consider the problem of minimizing $P(T > 1)$. Since it is equally likely for II to start either one or two places clockwise of I, this is equivalent to the problem of finding $p^\top = (p_1, p_2, p_3)$, with $p \geq 0$ and $p_1 + p_2 + p_3 = 1$, to minimize

$$p^\top \bar{B}_1 p = p^\top \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix} p = p_1^2 + p_2^2 + p_3^2 + p_1p_2 + p_2p_3 + p_3p_1\,.$$

It is not hard to show that this is minimized by $p_1 = p_2 = p_3 = \frac{1}{3}$ and that $P(T > 1)$ is minimized to $\frac{2}{3}$.

The minimum of $P(T > 2)$

Again suppose that II starts one location clockwise of I. Then the indicator matrix for not meeting within 2 steps is

$$B_2 := B_1 \otimes B_1 = \begin{pmatrix} B_1 & B_1 & 0 \\ 0 & B_1 & B_1 \\ B_1 & 0 & B_1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

Rows 1–9 (and columns 1–9) correspond respectively to Player I (or II) playing patterns of moves over the first two steps of $aa, ab, ac, ba, bb, bc, ca, cb, cc$.

The probability of not meeting within 2 steps is $p^\top \bar{B}_2 p$.

AW minimizes $P(T > 2)$

We find that $p^\top \bar{B}_2 p$ is minimized by $p^\top = \frac{1}{3}(1,0,0,0,0,1,0,1,0)$, with $p^\top \bar{B}_2 p = \frac{1}{3}$. I.e., '$aa$', '$bc$' and '$cb$' are to be chosen equally likely. This is the **AW** strategy.

$$\bar{B}_2 = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} & 1 & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix}$$

Another optimal strategy is $p^\top = (0,1,0,1,0,0,0,0,1)$, where '$ab$', '$ba$' and '$cc$' are to be chosen equally likely.

A quadratic programming problem

To prove that **AW** minimizes $p^\top \bar{B}_2 p$ we must solve a difficult quadratic programming problem.

The difficulty arises because $\bar{B}_2$ is not positive semidefinite. It's eigenvalues are $\{4, 1, 1, 1, 1, 1, 1, -\frac{1}{4}, -\frac{1}{4}\}$. This means that there can be local minima to $p^\top \bar{B}_2 p$.

E.g., $p = \frac{1}{9}(1, 1, 1, 1, 1, 1, 1, 1, 1)$, is a local minimum, with $p^\top \bar{B}_2 p = \frac{4}{9}$. However, this is not a global minimum.

In general, if a matrix C is not positive semidefinite, the following problem is NP-hard:

$$\text{minimize } \; p^\top C p \; : \; p \geq 0 \, , \; 1^\top p = 1 \, .$$

A method for finding lower bounds

Suppose we are trying to minimize $p^\top Cp$, but C is not positive semidefinite.

We can obtain a lower bound on the solution as follows.

$$
\begin{aligned}
&\min\{p^\top Cp \,:\, p \geq 0\,,\ 1^\top p = 1\} \\
&\qquad = \min\{\mathrm{trace}(Cpp^\top) \,:\, p \geq 0\,,\ 1^\top p = 1\} \\
&\qquad \geq \min\{\mathrm{trace}(CX) \,:\, X \succeq 0\,,\ X \geq 0\,,\ \mathrm{trace}(JX) = 1\}\,,
\end{aligned}
$$

where $J = 11^\top$ is a matrix of all 1s.

This is by using the fact that if p satisfies the l.h.s. constraints, then $X = pp^\top$ satisfies the r.h.s. constraints.

Semidefinite programming problems

Given symmetric matrices $C, A_1, \dots, A_m$, consider the problem

$$\text{minimize}\,\{\text{trace}(CX)$$
$$:\ X \succeq 0\,,\ X \geq 0\,,\ \text{trace}(A_i X) = b_i,\ i = 1, \dots, m\}\,.$$

This is a *Semidefinite Programming Problem* (SDP).

The minimization is over the components of X.
This can mean lots of decision variables.
If X is $j \times j$ and symmetric, then there are $j(j-1)/2$ variables.

SDPs can be solved to any degree of numerical accuracy using interior point algorithms (e.g., using Matlab and sedumi).

Semidefinite programming has been called *'linear programming for the 21st century'*. The SDP above becomes a linear programming problem in the special case that $C, A_1, \dots, A_m$ are all diagonal.

A lower bound on $p^\top \bar{B}_2 p$

As a relaxation of the quadratic program:

$$\text{minimize}\,\{p^\top \bar{B}_2 p \,:\, p \geq 0\,,\; 1^\top p = 1\}\,,$$

we consider the SDP:

$$\text{minimize}\,\{\text{trace}(\bar{B}_2 X) \,:\, X \succeq 0\,,\; X \geq 0\,,\; \text{trace}(J_2 X) = 1\}\,,$$

where J_2 is the 9×9 matrix of 1s. There are 36 decision variables. We find that the minimum value is $1/3$.

But $p^\top \bar{B}_2 p = 1/3$ for $p^\top = \frac{1}{3}(1, 0, 0, 0, 0, 1, 0, 1, 0)$.

So we may conclude that $1/3$ is the minimal value of $p^\top \bar{B}_2 p$.

Lower bounds on $P(T > k)$

Let ϕ_k be the minimal possible value of $P(T > k)$.

The problem of minimizing $P(T > k)$ is equivalent to minimizing $p^\top \bar{B}_k p$, where

$$B_k = B_1 \otimes B_{k-1}\,.$$

To find a lower bound on ϕ_k we consider the SDP:

$$\mathrm{minimize}\{\mathrm{trace}(\bar{B}_k X)\;:\;X \succeq 0\,,\;X \geq 0\,,\;\mathrm{trace}(XJ_k) = 1\}\,.$$

where J_k is a $3^k \times 3^k$ matrix of 1s.

Notice that $\bar{B}_k$ is $3^k \times 3^k$ and so the number of decision variables in the SDP grows very rapidly, as $\Omega(3^{2k})$.

Lower bounds on $E[\min\{T, k+1\}]$

Similarly, let w_k be the minimal possible value of the 'expected $k-$truncated rendezvous time', $E[\min\{T, k+1\}]$. Now

$$E[\min\{T, k+1\}] = p^\top M_k p \,,$$

where

$$M_k = J_k + B_1 \otimes J_{k-1} + \cdots + B_k \,.$$

To find a lower bound on w_k we consider the SDP:

$$\text{minimize}\,\{\text{trace}(\bar{M}_k X)\;:\; X \succeq 0\,,\; X \geq 0\,,\; \text{trace}(X J_k) = 1\}\,.$$

Lower bounds on ϕ_k and w_k

Recall that ϕ_k and w_k are, respectively, the infimal possible values of $P(T > k)$ and $E[\min\{T, k+1\}]$. Solving SDPs, we get

Lower bounds when players have a common clockwise:

k	1	2	3	4
ϕ_k	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{10}$	$\frac{1}{11}$
w_k	$\frac{5}{3}$	2	$\frac{20}{9}$	$\frac{21}{9}$

Lower bounds when players have no common clockwise:

k	1	2	3	4	5
ϕ_k	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{9}$	0.108697	
w_k	$\frac{5}{3}$	2	$\frac{20}{9}$	$\frac{21}{9}$	$\frac{65}{27}$ ‡

Observations

1. It is computationally infeasible to go much further. The number of decision variables in the SDP is 3240 when $k = 4$. For $k = 5$ it would be 29403.
2. These lower bounds prove that **AW** minimizes $E[\min\{T, k+1\}]$ as far as $k = 4$, and **AW** minimizes $P(T > k)$ as far as $k = 2$.
3. For $k = 3$ the lower bound on $p^\top \bar{B}_3 p$ is $2/10$, and this cannot be achieved. In fact, the minimum is $2/9$.
4. The answers depend on whether or not the players are provided with a common notion of clockwise.
   If they are not, then they must choose p such that 'b' and 'c' are treated the same. This requires $p = S_k p$, where S_k is a matrix that swaps the roles of the non-home locations.

A conjecture concerning **AW**

$$ET = \sum_{k=0}^{\infty} P(T > k)\,.$$

AW does not minimize every term in this sum. For example, with **AW** we get $P(T > 4) = \frac{1}{9}$, but there is another strategy for which $P(T > 4) = \frac{1}{10}$.

w_k is the minimal value of $E[\min\{T, k+1\}] = \sum_{j=0}^{k} P(T > j)$. It is found by minimizing $p^\top M_k p$, where

$$M_k = J_k + B_1 \otimes J_{k-1} + \cdots + B_k\,.$$

Empirically, we notice that the lower bounds for w_k are always achieved by **AW**, and are the same whether or not the players have a common notion of clockwise. This leads us to conjecture the following theorem.

The optimality of **AW** for K_3

Theorem 4 *The Anderson–Weber strategy is optimal for the symmetric rendezvous search game on K_3, minimizing $E[\min\{T, k+1\}]$ to w_k for all $k = 1, 2, \dots$, where*

$$w_k = \begin{cases} \frac{5}{2} - \frac{5}{2} 3^{-\frac{k+1}{2}}, & \text{when } k \text{ is odd,} \\ \frac{5}{2} - \frac{3}{2} 3^{-\frac{k}{2}}, & \text{when } k \text{ is even.} \end{cases}$$

Consequently, the minimal achievable value of ET is $w = \frac{5}{2}$.

$\{w_k\}_0^\infty = \{1, \frac{5}{3}, 2, \frac{20}{9}, \frac{21}{9}, \frac{65}{27}, \dots\}$.

Proof that **AW** is optimal on K_3

We begin by describing how we might prove that a given strategy minimizes $E[\min\{T,3\}] = P(T>0) + P(T>1) + P(T>2)$, or equivalently, that a given p minimizes $p^\top \bar{M}_2 p$.

1. Suppose we are trying to minimize $p^\top \bar{M}_2 p$, but $\bar{M}_2$ is not positive semidefinite.
2. Suppose we can find a matrix H_2, which is positive semidefinite and such that $M_2 \geq H_2$.
3. Suppose we can minimize $p^\top \bar{H}_2 p$. This provides a lower bound on the minimum of $p^\top \bar{M}_2 p$.
4. If this lower bound can be achieved, i.e., $p^\top(\bar{M}_2 - \bar{H}_2)p = 0$, then p minimizes $p^\top \bar{M}_2 p$.

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The minimum of $E[\min\{T, 3\}]$

We can take $p^\top = \frac{1}{3}(1,0,0,0,0,1,0,1,0)$ and

$$M_2 = \begin{pmatrix} 3 & 3 & 2 & 3 & 3 & 2 & 1 & 1 & 1 \\ 2 & 3 & 3 & 2 & 3 & 3 & 1 & 1 & 1 \\ 3 & 2 & 3 & 3 & 2 & 3 & 1 & 1 & 1 \\ 1 & 1 & 1 & 3 & 3 & 2 & 3 & 3 & 2 \\ 1 & 1 & 1 & 2 & 3 & 3 & 2 & 3 & 3 \\ 1 & 1 & 1 & 3 & 2 & 3 & 3 & 2 & 3 \\ 3 & 3 & 2 & 1 & 1 & 1 & 3 & 3 & 2 \\ 2 & 3 & 3 & 1 & 1 & 1 & 2 & 3 & 3 \\ 3 & 2 & 3 & 1 & 1 & 1 & 3 & 2 & 3 \end{pmatrix}$$

$$\geq H_2 = \begin{pmatrix} 3 & 3 & 2 & 3 & 3 & 2 & 1 & 1 & 0 \\ 2 & 3 & 3 & 2 & 3 & 3 & 0 & 1 & 1 \\ 3 & 2 & 3 & 3 & 2 & 3 & 1 & 0 & 1 \\ 1 & 1 & 0 & 3 & 3 & 2 & 3 & 3 & 2 \\ 0 & 1 & 1 & 2 & 3 & 3 & 2 & 3 & 3 \\ 1 & 0 & 1 & 3 & 2 & 3 & 3 & 2 & 3 \\ 3 & 3 & 2 & 1 & 1 & 0 & 3 & 3 & 2 \\ 2 & 3 & 3 & 0 & 1 & 1 & 2 & 3 & 3 \\ 3 & 2 & 3 & 1 & 0 & 1 & 3 & 2 & 3 \end{pmatrix}.$$

The eigenvalues of $\bar{M}_2$ are $\{19, \frac{5}{2}, \frac{5}{2}, 1, 1, 1, 1, -\frac{1}{2}, -\frac{1}{2}\}$, so it is not positive semidefinite. However, $\bar{H}_2$ is positive semidefinite, with eigenvalues $\{18, 3, 3, \frac{3}{2}, \frac{3}{2}, 0, 0, 0, 0\}$. Here

$$\bar{H}_2 p = \begin{pmatrix} 3 & \frac{5}{2} & \frac{5}{2} & 2 & \frac{3}{2} & \frac{3}{2} & 2 & \frac{3}{2} & \frac{3}{2} \\ \frac{5}{2} & 3 & \frac{5}{2} & \frac{3}{2} & 2 & \frac{3}{2} & \frac{3}{2} & 2 & \frac{3}{2} \\ \frac{5}{2} & \frac{5}{2} & 3 & \frac{3}{2} & \frac{3}{2} & 2 & \frac{3}{2} & \frac{3}{2} & 2 \\ 2 & \frac{3}{2} & \frac{3}{2} & 3 & \frac{5}{2} & \frac{5}{2} & 2 & \frac{3}{2} & \frac{3}{2} \\ \frac{3}{2} & 2 & \frac{3}{2} & \frac{5}{2} & 3 & \frac{5}{2} & \frac{3}{2} & 2 & \frac{3}{2} \\ \frac{3}{2} & \frac{3}{2} & 2 & \frac{5}{2} & \frac{5}{2} & 3 & \frac{3}{2} & \frac{3}{2} & 2 \\ 2 & \frac{3}{2} & \frac{3}{2} & 2 & \frac{3}{2} & \frac{3}{2} & 3 & \frac{5}{2} & \frac{5}{2} \\ \frac{3}{2} & 2 & \frac{3}{2} & \frac{3}{2} & 2 & \frac{3}{2} & \frac{5}{2} & 3 & \frac{5}{2} \\ \frac{3}{2} & \frac{3}{2} & 2 & \frac{3}{2} & \frac{3}{2} & 2 & \frac{5}{2} & \frac{5}{2} & 3 \end{pmatrix} \begin{pmatrix} \frac{1}{3} \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{3} \\ 0 \\ \frac{1}{3} \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \end{pmatrix}.$$

Thus p satisfies a Kuhn-Tucker condition for there to be a local minimum of $p^\top \bar{H}_2 p = 2$.
Since $\bar{H}_2 \succeq 0$, a local minimum is also a global minimum.
So $w_2 = 2$. This is achieved by **AW**.

Minimizing $E[\min\{T, k+1\}]$

Similarly, consider the problem of minimizing $E[\min\{T, k+1\}]$.

This is equivalent to minimizing $p^\top \bar{M}_k p$, where

$$M_k = J_k + B_1 \otimes J_{k-1} + \cdots + B_k\,.$$

As we did with H_2 for M_2, we look for H_k, such that $H_k \leq M_k$ and $\bar{H}_k \succeq 0$. This is a semidefinite programming problem

Notice that H_k is $3^k \times 3^k$ and so the number of decision variables grows very rapidly, as $\Omega(3^{2k})$. So there is a limit to what we can discover numerically.

How can we find H_k?

We can numerically solve the SDP:

$$\text{maximize}\{\text{trace}(J_2 H_2)\ :\ H_2 \le M_2,\ \bar{H}_2 \succeq 0\}\,.$$

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$$H_2 = \begin{pmatrix} 3.0000 & 2.7951 & 1.8324 & 2.8005 & 2.8005 & 2.0000 & 0.8857 & 1.0000 & 0.8857 \\ 1.8324 & 3.0000 & 2.7951 & 2.0000 & 2.8005 & 2.8005 & 0.8857 & 0.8857 & 1.0000 \\ 2.7951 & 1.8324 & 3.0000 & 2.8005 & 2.0000 & 2.8005 & 1.0000 & 0.8857 & 0.8857 \\ 0.8857 & 1.0000 & 0.8857 & 3.0000 & 2.7951 & 1.8324 & 2.8005 & 2.8005 & 2.0000 \\ 0.8857 & 0.8857 & 1.0000 & 1.8324 & 3.0000 & 2.7951 & 2.0000 & 2.8005 & 2.8005 \\ 1.0000 & 0.8857 & 0.8857 & 2.7951 & 1.8324 & 3.0000 & 2.8005 & 2.0000 & 2.8005 \\ 2.8005 & 2.8005 & 2.0000 & 0.8857 & 1.0000 & 0.8857 & 3.0000 & 2.7951 & 1.8324 \\ 2.0000 & 2.8005 & 2.8005 & 0.8857 & 0.8857 & 1.0000 & 1.8324 & 3.0000 & 2.7951 \\ 2.8005 & 2.0000 & 2.8005 & 1.0000 & 0.8857 & 0.8857 & 2.7951 & 1.8324 & 3.0000 \end{pmatrix}$$

and $\min_p\{p^\top H_2 p\} = 1.9999889$.

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and $\min_p\{p^\top H_2 p\} = 1.9999889$. But $\min_p\{p^\top H_2 p\} = 2$ using

$$H_2 = \begin{pmatrix} 3 & 3 & 2 & 3 & 3 & 2 & 1 & 1 & 0 \\ 2 & 3 & 3 & 2 & 3 & 3 & 0 & 1 & 1 \\ 3 & 2 & 3 & 3 & 2 & 3 & 1 & 0 & 1 \\ 1 & 1 & 0 & 3 & 3 & 2 & 3 & 3 & 2 \\ 0 & 1 & 1 & 2 & 3 & 3 & 2 & 3 & 3 \\ 1 & 0 & 1 & 3 & 2 & 3 & 3 & 2 & 3 \\ 3 & 3 & 2 & 1 & 1 & 0 & 3 & 3 & 2 \\ 2 & 3 & 3 & 0 & 1 & 1 & 2 & 3 & 3 \\ 3 & 2 & 3 & 1 & 0 & 1 & 3 & 2 & 3 \end{pmatrix}.$$

How to construct H_k

Let us search for H_k of a special form. For $i = 0, \dots, 3^k - 1$ we write $i_{\text{base}\,3} = i_1 \cdots i_k$ (keeping k digits, including leading 0s); so $i_1, \dots, i_k \in \{0, 1, 2\}$. Define

$$P_i = P_{i_1 \cdots i_k} = P_1^{i_1} \otimes \cdots \otimes P_1^{i_k} \,,$$

where

$$P_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} .$$

Observe that $M_k = \sum_i m_k(i) P_i$, where m_k is the first row of M_k. This motivates seeking H_k of the form

$$H_k = \sum_{i=0}^{3^k - 1} x_k(i) P_i \,.$$

Concluding steps of the proof

- The condition $M_k \geq H_k$ is equivalent to $m_k \geq x_k$.
- The matrices $P_0, \dots, P_{3^k-1}$ commute and so have a common set of eigenvectors.
- Let $\omega = -\frac{1}{2} + i\frac{1}{2}\sqrt{3}$, a cube root of 1. Then

$$V_k = V_1 \otimes V_{k-1}\,, \quad \text{where } V_1 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}.$$

- Let $V_k = U_k + iW_k$.
  Columns of V_k are eigenvectors of the P_i and also of M_k.
- Columns of U_k are eigenvectors of the $\bar{P}_i$ and also of $\bar{M}_k$.
- The condition $\bar{H}_k \succeq 0$ is equivalent to $U_k x_k \geq 0$.

We show that we may take $H_k = \sum_i x_k(i) P_i$, where

$$x_1 = (2,2,1)^\top \quad x_2 = (3,3,2,3,3,2,1,1,0)^\top$$

and choose a_k so that for $k \geq 3$,

$$\begin{aligned} x_k = & 1_k + (1,0,0)^\top \otimes x_{k-1} \\ & + (0,1,0)^\top \otimes (a_k, a_k, 2, 2, a_k, 2, 1, 1, 1)^\top \otimes 1_{k-3}\,. \end{aligned}$$

Here a_k is chosen maximally such that $U_k x_k \geq 0$ and $m_k \geq x_k$.

All rows of H_k have the same sum, and so $p^\top H_k p$ is minimized by $p = (1/3^k) 1_k$, and the minimum value is $p^\top H_k p = 1_k^\top x_k / 3^k$. So the theorem is true provided $1^\top x_k = 3^k w_k$. We have

$$1_k^\top x_k = 3^k + 1_{k-1}^\top x_{k-1} + 3^{k-2}(3 + a_k)\,.$$

So $1^\top x_k = 3^k w_k$ iff the above works when taking

$$a_k = \begin{cases} 3 - \dfrac{1}{3^{(k-3)/2}}, & \text{when } k \text{ is odd,} \\ 3 - \dfrac{2}{3^{(k-2)/2}}, & \text{when } k \text{ is even.} \end{cases}$$

Note that a_k increases monotonically in k, from 2 towards 3. As $k \to \infty$ we find $a_k \to 3$ and $1_k^\top x_k / 3^k \to \frac{5}{2}$.
Finally, we prove that with these a_k we have always have

1. $m_k \geq x_k$, (implying $M_k \geq H_k$).
2. $U_k x_k \geq 0$, (implying $\bar{H}_k \succeq 0$).

Both are proved by induction. The first is easy and the second is hard. To prove the second we use the recurrence relation for x_k to find recurrences relations for components of the vectors $U_k x_k$, and then show that all components are nonnegative. ∎

A convexity result for $P(T > k)$

Interestingly, there is an alternative, and purely probabilistic, proof that minimal value of $E[\min\{T, 3\}]$ is 2. It is a corollary of the following lemma.

Lemma 1 *Consider any rendezvous search game in which, given positive constants* $\{\alpha_1, \ldots, \alpha_k\}$*, the players' aim is to minimize*

$$\sum_{j=1}^{k} \alpha_i P(T > j)\,.$$

Then under any optimal strategy $\phi_j = P(T > j)$ *is convex decreasing in* j*, i.e.,*

$$\phi_{j+2} - \phi_{j+1} \geq \phi_{j+1} - \phi_j\,, \quad \textit{for all } j = 0, \ldots, k-2.$$

Proof. Let I_j denote the event that the players meet at their jth step, irrespective of any meeting previously. An optimal strategy minimizes

$$\sum_{j=1}^{k} \alpha_i P(T > j) = \sum_{j=1}^{k} \alpha_j \Big(1 - P(I_1 \cup \cdots \cup I_j)\Big) .$$

Since no improvement is possible if both players swap the locations they visit as steps $j+1$ and $j+2$, we must have

$$P(I_1 \cup \cdots \cup I_j \cup I_{j+1}) \geq P(I_1 \cup \cdots \cup I_j \cup I_{j+2}) .$$

Thus,

$$\begin{aligned}
\phi_{j+2} &= 1 - P(I_1 \cup \cdots \cup I_j \cup I_{j+1} \cup I_{j+2}) \\
&= 1 - P(I_1 \cup \cdots \cup I_j \cup I_{j+1}) - P(I_1 \cup \cdots \cup I_j \cup I_{j+2}) \\
&\qquad + P\big((I_1 \cup \cdots \cup I_j \cup I_{j+1}) \cap (I_1 \cup \cdots \cup I_j \cup I_{j+2})\big) \\
&\geq 1 - 2P(I_1 \cup \cdots \cup I_j \cup I_{j+1}) + P(I_1 \cup \cdots \cup I_j) \\
&= 2\phi_{j+1} - \phi_j .
\end{aligned}$$

Corollaries

In the particular case of $\alpha_1 = \cdots = \alpha_k = 1$ the players seek to minimize the expected k–truncated rendezvous time.

If they are playing the symmetric rendezvous game on K_3 and $k = 3$, we have that under the optimal policy,

$$\begin{aligned} E[\min\{T,3\}] &= P(T>0)+P(T>1)+P(T>2) \\ &\geq P(T>0)+P(T>1)+\big[2P(T>1)-P(T>0)\big] \\ &= 3P(T>1) \\ &\geq 2\,. \end{aligned}$$

But we have not be able to find a similar neat way to give a tight lower bound on $E[\min\{T,k\}]$ for any $k > 3$.

Symmetric rendezvous search on K_n

For K_3 we have seen that **AW** minimizes $E[\min\{T, k\}]$ for all $k = 1, 2, \dots$.

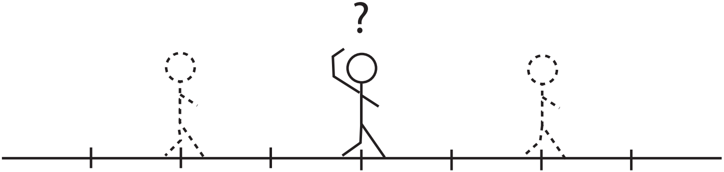
This is not true for K_4. However, **AW** does minimize $E[\min\{T, 3\}]$ and $E[\min\{T, 6\}]$.

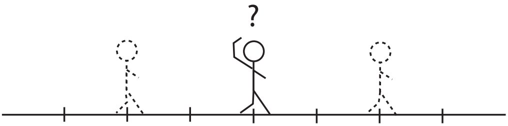
Similarly, **AW** minimizes $E[\min\{T, 4\}]$ for search on K_5.

This suggests the conjecture that for search on K_n, **AW** minimizes ET and also $E[\min\{T, k\}]$, for all k that are divisible by $n - 1$.

Symmetric rendezvous search on the line

Two players are placed 2 units apart on a line, randomly facing left or right. At each step each player must either move one unit forward or backwards. Each player knows that the other player is equally likely to be in front or behind him, and equally likely to be facing either way. How can they meet in the least expected time?



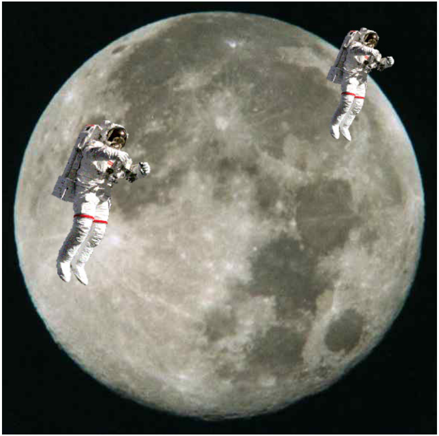


- $4.1820 \leq w \leq 4.2574$, and it is conjectured that $w = 4.25$.
- It seems not to help if the players are initially faced the same way.
  This is similar to the fact that on K_3 it is no help for players to have a common notion of clockwise.
- In the asymmetric version of rendezvous search, Players I and II may adopt different strategies. It is known that the minimal expected rendezvous time is 3.25, one less that the conjectured symmetric rendezvous value.

  Note that in the asymmetric version of rendezvous on K_3, the minimal expected rendezvous time is 1.5, also one less than the symmetric rendezvous value.

Symmetric rendezvous search in other spaces

Alpern (1976) has also proposed the following problem.



Two astronauts land at random spots on a planet (which is assumed to be a uniform sphere, without any known distinguishing marks or directions) How should they move so as to be within 1 kilometre of one another in the least expected time?