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# An Optimal Strategy in Multi-server Stochastic Scheduling

By Richard R. Weber and Peter Nash

*Cambridge University*

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Summary

Identical components are available for use in a piece of machinery. The number of components needed to operate the machine is a function of time and the lifetime of each component is described by a known probability distribution. Once a certain number of components have failed there will not be enough left to operate the machine. We find a strategy which for certain lifetime distributions delays this occurrence for as long as possible.

*Keywords*: STOCHASTIC SCHEDULING; MULTI-SERVER; STOCHASTIC ORDER; LADY'S NYLON STOCKING PROBLEM

## 1. Introduction

Suppose that $n$ identical components are available for use in a piece of machinery and that the number required to operate the machine is a function of time, $r(t)$. Components fail when in use according to a known probability distribution, and once a certain number have failed there will not be enough left to operate the machine. We are interested in scheduling the use of the components so as to delay this occurrence for as long as possible. Cox (1959) introduced this problem in connection with women's nylon stockings and $r(t) = 2$. It has been investigated by El-Sayyad (1967), Gait (1972), Nash (1973) and Glazebrook (1976). Cox suggested the intuitively reasonable strategy of always using those components with the greatest life expectancy. In this paper we show that if the component lifetime distribution has any monotone hazard rate then Cox's strategy is optimal in the strong sense of stochastic order. It maximizes for all $s$ the probability that we can operate the machine until at least time $s$.

Glazebrook and Nash (1976) considered the case of lifetime distributions with decreasing hazard rate and presented a fallacious proof of the optimality of Cox's strategy. The error is in a statement that is essential to the proof. It occurs in the third paragraph of p. 70 in the sentence which begins, "another way of stating (4) is . . .". This statement makes the false assumption that if two random functions $x_1(t)$ and $x_2(t)$ take values in $(0,1,2)$, are non-decreasing in $t$ $(1 \leqslant t < \infty)$, and are such that $P\{x_1(t) \leqslant z\} \leqslant P\{x_2(t) \leqslant z\}$ for all $t$ $(1 \leqslant t < \infty)$ and $z$ $(z = 0,1,2)$, then realizations of $x_1(t)$ and $x_2(t)$ can be matched in pairs so that for each pair we have $x_1(t) \leqslant x_2(t)$ for all $t$ $(1 \leqslant t < \infty)$.

The proof in this paper uses a new technique which has been used by Weber (1978) in solving a problem in the assignment of customers to a number of identical parallel servers.

## 2. An Optimal Scheduling Strategy

The following assumptions hold throughout this section. Time proceeds in discrete steps, $0, 1, 2, \ldots$. "Interval $t$" is the interval $(t, t+1)$. At the beginning of interval 0 there are available $n$ components, $c_1, c_2, \ldots, c_n$. During interval $t$ the number of components required to operate the machine is a constant, given by the *usage function* $r(t)$ $(t = 0, 1, 2, \ldots)$. The usage function is non-decreasing in $t$. Components fail randomly during use; each has the same known lifetime distribution when new, but different components may have had different amounts of use prior to interval 0.

With each component is associated an *age*, which is the amount of usage it has received. The age of $c_i$ at time $t$ is denoted by $x_i(t)$. The *hazard rate* for a usable component is defined to be the probability that it fails in its next interval of use. The hazard rate for a component of age $x$ is denoted by $p(x)$ $(=1-q(x))$. If $c_i$ is used during interval $t$ it fails with probability $p(x_i(t))$; if it does not fail then its age at time $t+1$ is $x_i(t)+1$. If $c_i$ is not used during interval $t$ then $x_i(t+1)=x_i(t)$. Although components will usually have integer ages we will sometimes allow their initial ages to take any values in $[0,\infty)$. The hazard rate function $p(x)$ is assumed to be a strictly monotone (increasing or decreasing), differentiable function of the age $x$.

A *state* of the components is the vector of their ages, denoted by $x$ $(=(x_1,x_2,\ldots,x_n))$. If $c_i$ has failed this is denoted by writing $*$ in the place of $x_i$. A *strategy* is a function which, for any value $r(t)$ of the usage function and any state with at least $r(t)$ serviceable components, determines a set of $r(t)$ components to be used during interval $t$.

We consider the problem of finding a strategy for scheduling the components which maximizes the probability that the machine can be operated until some fixed time horizon. $S^*$ denotes the strategy which schedules for use during interval $t$ those $r(t)$ components whose life expectancies are greatest. For strictly monotone hazard rates $S^*$ can be realized by scheduling for use those components whose hazard rates are smallest. This follows by writing the expected remaining lifetime of a component of age $x$ as

$$\sum_{i=0}^{\infty}\prod_{j=0}^{i} q(x+j).$$

When two or more components have equal hazard rates they have the same age and so are statistically identical. Strategies which differ only in scheduling amongst components of equal hazard rate are therefore equivalent and there is no essential ambiguity in this realization of $S^*$.

Denote by $P^S\{r(\cdot),s,x\}$ the probability that we can operate the machine until at least the end of some fixed interval $s^*$, given that we start at time $s$ in state $x$ and apply scheduling strategy $S$ with usage function $r(\cdot)$. In the special case where the scheduling strategy is $S^*$ we use $P\{r(\cdot),s,x\}$ in place of $P^{S^*}\{r(\cdot),s,x\}$. Define

$$P_i^S\{r(\cdot),s,x\}=p(x_i)P^S\{r(\cdot),s,(x_1,\ldots,*,\ldots,x_n)\}+q(x_i)P^S\{r(\cdot),s,(x_1,\ldots,x_i+1,\ldots,x_n)\}.$$

Again we write $P_i\{r(\cdot),s,x\}$ in the place of $P_i^{S^*}\{r(\cdot),s,x\}$. Denote by $A$ the set of states for which every component either has failed or is of integer age.

*Theorem* 1. $S^*$ is optimal in the sense that for any other strategy $S$,

$$P\{r(\cdot),s,x\}\geqslant P^S\{r(\cdot),s,x\}$$

for all $r(\cdot)$, $s$ and $x\in A$.

*Proof.* The proof is by induction on $s$. Clearly the theorem is true for $s=s^*$. Assume it is true for $s=t+1,\ldots,s^*$. This implies that it is optimal to follow $S^*$ from interval $t+1$ onwards. We need to show that it is optimal to follow $S^*$ during interval $t$. We do this by considering two strategies, $S_i$ and $S_j$, which are both identical to $S^*$ from $t+1$ onwards. During interval $t$, both schedule for use the same set $C$ of $r(t)-1$ components. They differ in that the $r(t)$th component scheduled by $S_i$ is $c_i$, while for $S_j$ it is $c_j$. We show that if $p\{x_i(t)\}<p\{x_j(t)\}$ then

$$P^{S_i}\{r(\cdot),s,x(t)\}\geqslant P^{S_j}\{r(\cdot),s,x(t)\}. \quad (1)$$

The result then follows, for if $S$ is any strategy identical with $S^*$ from $t+1$ onwards then $S^*$ can be obtained from $S$ by successively replacing components scheduled by $S$ at time $t$ by components of smaller hazard rate.

To prove (1) we establish a correspondence between $P^{S_i}\{r(\cdot), t, x\}$ and $P_i^S\{\hat{r}(\cdot), t, x\}$ for a suitably chosen strategy $S$, where

$$\hat{r}(t) = r(t)-1, \quad \hat{r}(s) = r(s), \; s = t+1, \ldots, s^*.$$

Define the strategy $S$ for the usage function $\hat{r}(\cdot)$ so that $S$ schedules the set of components $C$ at time $t$ and is identical with $S^*$ (and hence $S_i$ and $S_j$) thereafter. Considering now what happens to $c_i$ under $S_i$, and noting that what happens to the remaining components is the same for $S, \hat{r}(\cdot)$ and the starting state $(x_1, \ldots, x_i+1, \ldots, x_n)$ as for $S_i, r(\cdot)$ and the starting state $(x_1, \ldots, x_i, \ldots, x_n)$, we have

$$P^{S_i}\{r(\cdot), t, x\} = P_i^S\{\hat{r}(\cdot), t, x\}.$$

This says that scheduling $c_i$ and the components of $C$ during interval $t$ under $S_i, r(\cdot)$ accomplishes the same thing as giving $c_i$ one interval of usage just prior to time $t$ and then scheduling the components of $C$ during interval $t$ under $S, \hat{r}(\cdot)$. A similar statement holds for $P^{S_j}\{r(\cdot), t, x\}$, so that

$$P^{S_i}\{r(\cdot), t, x\} - P^{S_j}\{r(\cdot), t, x\} = P_i^S\{\hat{r}(\cdot), t, x\} - P_j^S\{\hat{r}(\cdot), t, x\}. \tag{2}$$

Now define

$$D_{i,j}\{r(\cdot), s, x\} = P_i\{r(\cdot), s, x\} - P_j\{r(\cdot), s, x\},$$

for any $r(\cdot), s, x$. In Theorems 2 and 3 we shall show that if $p(x_i) < p(x_j)$, then

$$D_{i,j}\{r(\cdot), s, x\} \geqslant 0, \tag{3}$$

for any $r(\cdot), s, x$. For the moment assume this is true. Let $X$ be the random state reached at the end of interval $t$ using strategy $S$ for $\hat{r}(\cdot)$. This means that for $c_k \notin C$ we have $X_k = x_k(t)$, since $S$ schedules only the components of $C$ at time $t$. For $c_k \in C$ we have $X_k = *$ or $x_k(t)+1$ as $c_k$ does or does not fail during interval $t$. Since $S$ and $S^*$ are identical from time $t+1$ onwards,

$$P_i^S\{\hat{r}(\cdot), t+1, X\} - P_j^S\{\hat{r}(\cdot), t+1, X\} = D_{i,j}\{\hat{r}(\cdot), t+1, X\}. \tag{4}$$

Since $\hat{r}(\cdot)$ is identical with $r(\cdot)$ from $t+1$ onwards,

$$D_{i,j}\{\hat{r}(\cdot), t+1, X\} = D_{i,j}\{r(\cdot), t+1, X\}. \tag{5}$$

Since $c_i \notin C$ we have $X_i = x_i(t)$, so that

$$P_i^S\{\hat{r}(\cdot), t, x\} = E_X[P_i^S\{\hat{r}(\cdot), t+1, X\}],$$

and similarly for $P_j^S\{r(\cdot), t, x\}$. Therefore

$$P_i^S\{\hat{r}(\cdot), t, x\} - P_j^S\{\hat{r}(\cdot), t, x\} = E_X[P_i^S\{\hat{r}(\cdot), t+1, X\} - P_j^S\{\hat{r}(\cdot), t+1, X\}].$$

Combining this last expression with (2), (4) and (5) we have

$$P^{S_i}\{r(\cdot), t, x\} - P^{S_j}\{r(\cdot), t, x\} = E_X[D_{i,j}\{r(\cdot), t+1, X\}] \tag{6}$$

If $p(x_i) < p(x_j)$, then since $X_i = x_i$ and $X_j = x_j$, we have from (3), that whatever the other components of $X$,

$$D_{i,j}\{r(\cdot), t+1, X\} \geqslant 0.$$

Applying this to the right-hand side of (6) the result is proved.

*Corollary*. Suppose that we start at time 0 in the state $x \in A$ with $n$ serviceable components. If the usage function is such that we can certainly operate the machine as long as less than $m$ components have failed, i.e. $r(s) \leqslant n-m+1$ for all $s$, then

(i) $S^*$ maximizes the probability that the $m$th failure occurs after interval $s$ for all $s$, and

(ii) $S^*$ maximizes the expected time until the $m$th failure.

*Proof.*

(i) Let $\bar{r}(t) = r(t)$ $(t = 0,1,2,\ldots,s)$, $s^* = s+1$ and $\bar{r}(s+1) = n-m+1$. Then $P^S\{\bar{r}(\cdot),0,x\}$ is the probability that the $m$th failure occurs after interval $s$ and is maximized when $S = S^*$ by Theorem 1.

(ii) The expected time to the $m$th failure using strategy $S$ is

$$\sum_{s=0}^{\infty} P\{m\text{th failure occurs after interval } s \text{ using strategy } S\}$$

Each term of this sum is maximized for $S = S^*$.

Theorem 1 and its corollary are the main results of this paper. To complete the proof of the theorem we show in Theorems 2 and 3 that for $x \in A$ and $p(x_i) < p(x_j)$ we have $D_{i,j}\{r(\cdot),s,x\} \geqslant 0$ for all $r(\cdot)$ and $s$. This is a result about $S^*$ and it is the subject of the rest of this section. From now on we are concerned only with scheduling which takes place under $S^*$. Let $C\{r(\cdot),s,x\}$ be the set of components scheduled by $S^*$ at time $s$ when we start at time $s$ in state $x$ and the usage function is $r(\cdot)$. Note that $C\{r(\cdot),s,x\}$ is a set of $r(s)$ components of smallest hazard rate, by the definition of $S^*$.

*Theorem* 2. Suppose that $p(x)$ is strictly increasing in $x$. Then $D_{i,j}\{r(\cdot),s,x\} \geqslant 0$ for all $r(\cdot), s$ and any $x$ with $x_i < x_j$. Moreover if $c_i, c_j \in C\{r(\cdot),s,x\}$ then

$$(d/dx_i)\, D_{i,j}\{r(\cdot),s,x\} \text{ exists and is } \leqslant 0. \tag{7}$$

*Proof.* The proof is by induction on $s$. Clearly the theorem is true for $s = s^*$. Assume it is true for $s = t+1,\ldots,s^*$. We show that this implies that it is true for $s = t$. Let $C = C\{r(\cdot),t,x\}$, and let $X$ be the random state of the components at the end of interval $t$ when we start interval $t$ in state $x$. There are three possibilities.

(i) $c_i, c_j \notin C$. In this case $X_i = x_i$ and $X_j = x_j$ so that $D_{i,j}\{r(\cdot),t,x\} = E_X[D_{i,j}\{r(\cdot),t+1,X\}]$. By the inductive hypothesis the theorem may be applied to $D_{i,j}\{r(\cdot),t+1,X\}$ and the result is proved.

(ii) $c_i \in C$, $c_j \notin C$. This case is more complicated. A mechanism is required for following the component which is scheduled in place of $c_i$, when it fails or is not scheduled when it is serviceable. Accordingly, let $k$ be a component of least hazard rate amongst those not in $C$. Define a random index $i(X)$ by

$$i(X) = \begin{cases} i & \text{if } c_i \text{ would be scheduled at time } t \text{ if the components were in} \\ & \text{state } (x_1,\ldots,X_i,\ldots,x_n). \\ k & \text{otherwise.} \end{cases}$$

Now recall the definition

$$P_i\{r(\cdot),t,x\} = p(x_i)P\{r(\cdot),t,(x_1,\ldots,x_{i-1},*,x_{i+1},\ldots,x_n)\} \\ +q(x_i)P\{r(\cdot),t,(x_1,\ldots,x_{i-1},x_i+1,x_{i+1},\ldots,x_n)\}.$$

Consider a starting state $(x_1,\ldots,x_{i-1},*,x_{i+1},\ldots,x_n)$ at time $t$. The components scheduled during interval $t$ are the members of $C$ except $c_i$, which has failed, plus $c_{i(X)}$ $(= c_k)$ which replaces $c_i$. Thus, at time $t+1$,

$$x_m(t+1) = X_m, \quad m \neq i(X),$$
$$x_{i(X)}(t+1) = \begin{cases} * & \text{if } c_{i(X)} \text{ fails in its first interval of usage.} \\ x_{i(X)}+1 & \text{otherwise.} \end{cases}$$

Hence

$$P\{r(\cdot),t,(x_1,\ldots,x_{i-1},*,x_{i+1},\ldots,x_n)\} = E_X[P_{i(X)}\{r(\cdot),t+1,X\} \mid X_i = *]. \tag{8}$$

Similarly,

$$P\{r(\cdot), t, (x_1, \ldots, x_{i-1}, x_i+1, x_{i+1}, \ldots, x_n)\} = E_X[P_{i(X)}\{r(\cdot), t+1, X\} \mid X_i = x_i+1]. \tag{9}$$

Thus,

$$P_i\{r(\cdot), t, x\} = E_X[P_{i(X)}\{r(\cdot), t+1, X\}]. \tag{10}$$

Since $c_j \notin C$ we have (as in case (i)),

$$P_j\{r(\cdot), t, x\} = E_X[P_j\{r(\cdot), t+1, X\}]. \tag{11}$$

Combining (10) and (11) we have,

$$D_{i,j}\{r(\cdot), t, x\} = E_X[D_{i(X),j}\{r(\cdot), t+1, X\}].$$

Now $c_i \in C$, $c_j \notin C$, and by the definition of $k$ we have $p(x_k) \leqslant p(x_j)$, so that $X_{i(X)} \leqslant X_k \leqslant X_j$ with probability 1. By the inductive hypothesis therefore, $D_{i(X),j}\{r(\cdot), t+1, X\} \geqslant 0$, with probability 1. Thus $D_{i,j}\{r(\cdot), t, x\} \geqslant 0$, and the theorem is proved for this case.

(iii) $c_i, c_j \in C$. In this case (3) follows from (7) by noting that if $x_i = x_j$ then $c_i$ and $c_j$ are identical so $D_{i,j}\{r(\cdot), t, x\} = 0$.

It remains to prove that (7) holds for $s = t$ when $c_i, c_j \in C$. Using the argument that led to (10) we have

$$\begin{aligned} D_{i,j}\{r(\cdot), t, x\} = p(x_i)\, &E_X[D_{i(X),j(X)}\{r(\cdot), t+1, X\} \mid X_i = *] \\ &+ q(x_i)\, E_X[D_{i(X),j(X)}\{r(\cdot), t+1, X\} \mid X_i = x_i+1]. \end{aligned} \tag{12}$$

There are two cases to consider.

(a) $x_i+1 > x_k$. Then $x_j+1 > x_k$ also, and so with probability 1, $i(X) = j(X) = k$. The right-hand side of (12) is then identically zero and (7) holds.

(b) $x_i+1 \leqslant x_k$. Differentiating (12) gives

$$\begin{aligned} (d/dx_i)\, D_{i,j}\{r(\cdot), t, x\} = p'(x_i)\, &E_X[D_{i(X),j(X)}\{r(\cdot), t+1, X\} \mid X_i = *] \\ &- p'(x_i)\, E_X[D_{i(X),j(X)}\{r(\cdot), t+1, X\} \mid X_i = x_i+1] \\ &+ q(x_i)\, E_X[(d/dx_i)\, D_{i(X),j(X)}\{r(\cdot), t+1, X\} \mid X_i = x_i+1], \end{aligned} \tag{13}$$

remembering that $q(x_i) = 1 - p(x_i)$ and noting that $E_X[D_{i(X),j(X)}\{r(\cdot), t+1, X\} \mid X_i = *]$ does not depend on $x_i$. Now $p'(x) > 0$ and $j(X)$ must be either $j$ or $k$. If $j(X) = j$ then $x_j+1 \leqslant x_k$, so that in either event $X_{j(X)} \leqslant X_k$. If $X_i = *$ then $i(X) = k$ and $X_{i(X)} \geqslant X_{j(X)}$, so that by the inductive hypothesis the first term on the right-hand side of (13) is $\leqslant 0$: if $X_i = x_i+1$ then $i(X) = i$ and $X_{i(X)} = x_i+1 \leqslant X_{j(X)}$, so that by the inductive hypothesis the second term on the right-hand side of (13) is $\leqslant 0$; since $r(\cdot)$ is non-decreasing, if $X_i = x_i+1$ then

$$c_{i(X)} = c_i, c_{j(X)} \in C\{r(\cdot), t+1\ \ X\}$$

so that by the inductive hypothesis the third term on the right-hand side of (13) is $\leqslant 0$. Thus $(d/dx_i)\, D_{i,j}\{r(\cdot), t, x\} \leqslant 0$ and the proof is complete.

*Theorem* 3. Suppose that $p(x)$ is strictly decreasing in $x$. Then $D_{i,j}\{r(\cdot), s, x\} \geqslant 0$, for all $r(\cdot)$, $s$, and $x$ with $x_i > x_j$ and such that every component which is not amongst the $r(s)-1$ of least hazard rate either has failed or is of integer age. Moreover, if $c_i, c_j \in C\{r(\cdot), s, x\}$ then $(d/dx_i)\, D_{i,j}\{r(\cdot), s, x\}$ exists and is $\geqslant 0$.

*Proof*. The proof is exactly analogous to that for Theorem 2. The added condition on the states $x$ is necessary to enable derivation of (10) and (11) to go through.

## 3. EXTENSIONS AND COUNTER-EXAMPLES

The class of lifetime distributions for which the results of Section 2 are true may easily be extended from those of strictly monotone hazard rate to those of monotone hazard rate. The proofs go through as before. A continuous time version of Theorem 1 follows as the limit of discrete time problems by approximating with arbitrarily small discrete intervals and then applying a continuity argument.

The following counter-examples to Corollary 1(ii) demonstrate that the assumptions of monotone hazard rate and non-decreasing usage function may not in general be relaxed.

(i) $S^*$ is not optimal for arbitrary lifetime distributions. Suppose that $p(0) = 0, p(1) = 0{\cdot}999$, $p(s) = 0{\cdot}4$ $(s = 2,3,4,\ldots)$, $x = (0,2,2)$ and $r(s) = 2$ $(s = 0,1,2,\ldots)$. Then $S^*$ schedules the two components of age 2 during interval 0. But the expected time until the second failure is approximately 1/40 less under $S^*$ than under the strategy which schedules the component of age 0 during interval 0 and then schedules as $S^*$ thereafter.

(ii) $S^*$ is not optimal for arbitrary $r(\cdot)$. Suppose that $p(0) = p(1) = \frac{1}{8}$, $p(2) = p(3) = \frac{1}{4}$, $p(s) = 1$ $(s = 4,5,6,\ldots)$, $x = (0,1,2,4,4,4,4,4)$, $r(0) = 1$, $r(1) = 3$, $r(2) = 1$ and $r(s) = 6$ $(s = 3,4,5,\ldots)$. Then $S^*$ schedules the component of age 0 during interval 0. But the expected time until the third failure is 7/4096 less under $S^*$ than under the strategy which schedules the component of age 1 during interval 0 and then schedules as $S^*$ thereafter.

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