SIAM J. APPL. MATH.
Vol. 44, No. 4, August 1984

015

INEQUALITIES AND BOUNDS IN STOCHASTIC SHOP SCHEDULING*

MICHAEL PINEDO† AND RICHARD WEBER‡

Abstract. In this paper, stochastic shop models with m machines and n jobs are considered. A job has to be processed on all m machines, while certain constraints are imposed on the order of processing. The effect of the variability of the processing times on the expected completion time of the last job (the makespan) and on the sum of the expected completion times of all jobs (the flow time) is studied. Bounds are obtained for the expected makespan when the processing time distributions are New Better (Worse) than Used in Expectation.

Key words. stochastic scheduling, flow shop, job shop, open shop, exponential distribution, makespan, flow time

1. Introduction. Consider a shop with m machines and n jobs. Any given job requires processing on each one of the m machines and all jobs are available for processing at $t=0$. The manner in which the jobs are routed through the system is predetermined and fixed and depends on the particular shop model under consideration. The processing time of job j on machine i is a random variable X_{ij} with distribution G_{ij}. The joint distribution of the random variables $X_{11}, \cdots, X_{mn}$ may have one of the following two forms:

(i) The mn processing times $X_{11}, \cdots, X_{mn}$ are mutually independent.

(ii) The m processing times of a job on the m machines are identical, but the processing time of any given job on a machine is independent of the processing time of any other job on that machine, i.e. $X_{1j} = \cdots = X_{mj} = X_j$ with distribution G_j for $j=1, \cdots, n$ and X_j and X_k are mutually independent if $j \neq k$.

In what follows, these two cases are called, respectively, the independent and the equal case. In this paper, the effect of the processing times' variability on the expected completion time of the last job (the makespan) and on the sum of the expected completion times of all jobs (the flow time) is studied.

Four shop models are considered, namely flow shops with an unlimited storage space in between the machines, flow shops with no storage space in between the machines, job shops and open shops. A short description of these models follows.

(I) *Flow shops with unlimited intermediate storage.* The n jobs are to be processed on the m machines with the order of processing on the different machines being the same for all jobs. Each job has to be processed first on machine 1, then on machine 2, etc. The sequence in which the jobs go through the system is predetermined; job 1 has to go first through the system, followed by job 2, etc. There is an infinite intermediate storage in between any two consecutive machines; if machine $i+1$ is busy when job j is completed on machine i, job j is stored in between machines i and $i+1$. Preemptions are not allowed and a job may not "pass" another job while waiting for a machine.

(II) *Flow shops with no intermediate storage.* This model is similar to the previous model. The only difference lies in the fact that now there is *no* intermediate storage

* Received by the editors March 8, 1983, and in revised form October 5, 1983. This research was supported in part by the Office of Naval Research under contract N00014-80-k-0709 and in part by the National Science Foundation under grant EC5-8115344.

† Department of Industrial Engineering and Operations Research, Columbia University, New York, New York 10027.

‡ Control and Management Systems Division, Engineering Department, Cambridge University, Cambridge, England.

in between the machines. This may have the following effect: Job j, after completing its processing on machine i, may not leave machine i if job $j-1$ is still being processed on machine $i+1$. Job $j+1$ cannot start then its processing on machine i. This phenomenon is called blocking.

(III) *Job shops.* Only job shops with two machines are considered. Some of the jobs, say jobs $1, \cdots, p$, have to be processed first on machine 1 and afterwards on machine 2 (job 1 going first, followed by job 2, etc.). The remaining q $(=n-p)$ jobs have to be processed first on machine 2 and afterwards on machine 1 (job $p+1$ going first, followed by job $p+2$, etc.). There is an unlimited intermediate storage in between the two machines, so no blocking will occur. The policy under which the jobs are to be processed on the two machines is predetermined and under this policy jobs $1, \cdots, p$ $(p+1, \cdots, n)$ must have completed their processing on machine 1 (2) before any one of jobs $p+1, \cdots, n$ $(1, \cdots, p)$ is allowed to start on machine 1 (2). It is clear that if p is either 0 or n, this job shop reduces to a two machine flow shop with unlimited intermediate storage.

(IV) *Open shops.* Only two machine open shops are considered. The order in which a job is to be processed on the two machines is now immaterial. There is an unlimited intermediate storage, so no blocking will occur. Only policies are considered which always give priority to jobs which have not yet received processing on either one of the two machines.

In the literature these models have been dealt with extensively. The research in the past has been aimed mainly at finding job sequences and policies that minimize criteria such as the expected makespan and the expected flow time. For a survey of these results, see Pinedo and Schrage (1982) and, more recently, Pinedo (1983). Milch and Waggoner (1972) studied the two machine job shop where the two processing times of any given job are independent exponentially distributed with mean one and obtained a closed form expression for the expected makespan.

A summary of the results follows. Section 2 discusses a form of stochastic dominance based on variability ordering. The effect of the processing times variability on the expected makespan and on the expected flow time is studied for the first, second and third models described above. In § 3, closed form expressions for the expected makespan are presented for the first three models when the processing times of any given job on the various machines are i.i.d. exponential with mean one. Furthermore, bounds are obtained for the expected makespan when the processing times of any given job on the various machines are independent and NBUE (NWUE) with mean one. Section 4 repeats the work of § 3 for the equal case. In § 5, the equal and the independent cases of the two machine open shop are considered. Again, closed form expressions are obtained when the processing time distributions are exponential with mean one and bounds are obtained for when they are NBUE (NWUE).

The following notation and terminology is used. $S_{m,c,k}$ denotes a shop. If the S is an F, the shop is a flow shop; if it is a J a job shop, and if it is an O an open shop. The subscript m denotes the number of machines. If the c is an i (e), then the processing times are distributed according to the independent (equal) case. The k indicates the size of the intermediate storage; it is omitted if the shop is an open shop or a job shop. The time job j leaves the system is denoted by C_j; the makespan and the flow time are respectively denoted by $C_{\max}$ and $\sum C_j$. The time epoch at which job j leaves machine i is denoted by T_{ij}. The makespan and flow time of shop $S_{m,c,k}$ are denoted by $C_{\max}(S_{m,c,k})$ and $\sum C_j(S_{m,c,k})$, respectively; if it is clear from the context which shop is being considered, the argument $S_{m,c,k}$ is omitted. When all processing time distributions are exponential with mean one, this is indicated by an asterisk, e.g., $C^*_{\max}(S_{m,c,k})$ or $C^*_{\max}$.

2. Preliminaries. The random variable Y_1 with distribution F_1 is said to be more variable than the random variable Y_2 with distribution F_2 if

$$\int_0^\infty h(x)\,dF_1(x) \geqq \int_0^\infty h(x)\,dF_2(x)$$

for all functions h that are increasing convex. This form of stochastic dominance has been used repeatedly in the literature (see Bessler and Veinott (1966), Stoyan and Stoyan (1969), Niu (1981), Whitt (1980) and their references) and is written as $Y_1 \underset{\vee}{>} Y_2$. If $E(Y_1)=E(Y_2)$, then Y_1 is more variable than Y_2 if and only if

$$\int_0^\infty h(x)\,dF_1(x) \geqq \int_0^\infty h(x)\,dF_2(x)$$

for all functions h which are convex, not necessarily increasing.

A random variable Y_1 is said to be NBUE (NWUE) if

$$E(Y_1-t \mid Y_1>t) \leqq (\geqq) E(Y_1) \quad \text{for all } t \geqq 0.$$

NBUE (NWUE) stands for New Better (Worse) than Used in Expectation.

LEMMA 1. *Let $E(Y_1)=E(Y_2)$ and let Y_1 be an exponential random variable. If Y_2 is* NBUE (NWUE), *then $Y_2 \underset{\vee}{<} (\underset{\vee}{>})\, Y_1$.*

Proof. See Marshall and Proshan (1972).

LEMMA 2. *Let $Y_i, Z_i, i=1,\cdots,n$ be independent random variables. Then $Y_i \underset{\vee}{<} Z_i$ for all $i=1,\cdots,n$ if and only if $h(Y_1,\cdots,Y_n) \underset{\vee}{<} h(Z_1,\cdots,Z_n)$ for all increasing convex functions.*

Proof. See Bessler and Veinott (1966). Consider the shop $F_{m,c,k}$, $m=2,3,\cdots$, $c=e, i$, $k=0,1,2,\cdots$, and the shop $J_{2,c}$, $c=e, i$.

LEMMA 3. *In the shops $F_{m,c,k}$ and $J_{2,c}$, the time epoch T_{ij}, the makespan $C_{\max}$ and the flow time ΣC_j are functions which are increasing convex in X_{ij}.*

Proof. Consider first $F_{m,i,0}$. For the first job that goes through the system the following holds.

$$T_{i1}=\sum_{l=1}^{i} X_{l1}, \qquad i=1,\cdots,m.$$

This is clearly an increasing convex function. For job j, $j=2,\cdots,n$, the following holds.

$$T_{1j}=\max(T_{1,j-1}+X_{1j},\,T_{2,j-1}), \qquad j=2,\cdots,n,$$

$$T_{i,j}=\max(T_{i-1,j}+X_{ij},\,T_{i+1,j-1}), \qquad i=2,\cdots,m, \quad j=2,\cdots,n.$$

It follows by induction that for $F_{m,i,0}$ the time epoch T_{ij}, the makespan $C_{\max}$ $(=T_{mn})$ and the flow time $\sum C_j$ are functions which are increasing convex in X_{ij}. The proof of the lemma for $F_{m,e,0}$ is similar.

The result for $F_{m,i,k}$ can be shown by assuming k dummy machines in between any two real machines. The processing times of the n jobs on a dummy machine are assumed to be zero. Note that with n jobs the shop $F_{m,i,n-1}$ behaves just like the shop $F_{m,i,\infty}$. The proof for $F_{m,e,k}$ is similar.

In $J_{2,i}$ job j, $j=2,\cdots,p$, starts its processing on machine 2 at $\max(T_{1j}, T_{2,j-1})$. Therefore

$$T_{2j}=\max(T_{1j},\,T_{2,j-1})+X_{2j}, \qquad j=2,\cdots,p.$$

Note that $T_{11}=X_{11}$ and

$$T_{21}=\max\left(X_{11},\ \sum_{j=p+1}^{n} X_{2j}\right)+X_{21}.$$

A similar expression can be formulated for the departure times of jobs $p+1,\cdots,n$ from machine 1. It follows then by induction that $T_{2j}, j=1,\cdots,p$, and $T_{1j}, j=p+1,\cdots,n$, are functions that are increasing convex in X_{ij}. Now $C_{\max}=\max(T_{2p}, T_{1n})$. This proves the lemma for $J_{2,i}$. Again, the proof of the lemma for $J_{2,e}$ is similar.

Now, consider two shops of the same type, the type being one of the $F_{m,c,k}$ shops or one of the $J_{2,c}$ shops. A distinction is made between these two shops through the use of a prime and a double prime, for example, $F'_{3,i,\infty}$ and $F''_{3,i,\infty}$. The processing time distributions in one shop are not identical to the processing time distributions in the other shop: The processing time of job j on machine i in the first (second) shop is denoted by X'_{ij} (X''_{ij}) and its distribution by G'_{ij} (G''_{ij}). All other quantities of interest in the two shops receive a prime and a double prime as well. In the subsequent theorem and corollaries shops $F'_{m,c,k}$ and $J'_{2,c}$ are compared with shops $F''_{m,c,k}$ and $J''_{2,c}$ respectively. The results follow immediately from Lemmas 1, 2 and 3, and are therefore presented without proofs.

THEOREM 1. *If*

$$X'_{ij} \underset{\smile}{<} X''_{ij}, \qquad i=1,\cdots,m, \quad j=1,\cdots,n,$$

then

$$T'_{ij} \underset{\smile}{<} T''_{ij}, \qquad i=1,\cdots,m, \quad j=1,\cdots,n,$$

$$C'_{\max} \underset{\smile}{<} C''_{\max}$$

and

$$\sum C'_j \underset{\smile}{<} \sum C''_j.$$

COROLLARY 1. *If*

$$E(X'_{ij})=E(X''_{ij})$$

and

$$X'_{ij} \underset{\smile}{<} X''_{ij}, \qquad i=1,\cdots,m, \quad j=1,\cdots,n,$$

then

$$E(C'_{\max}) \leqq E(C''_{\max})$$

and

$$E(\sum C'_j) \leqq E(\sum C''_j).$$

Now, let all the processing times in $F'_{m,c,k}$ and $J'_{2,c}$ be exponentially distributed with mean one and replace the prime by an asterisk, i.e. $F^*_{m,c,k}$ and $J^*_{2,c}$. Let the processing times in $F''_{m,c,k}$ and $J''_{2,c}$ have mean one and be either all NBUE or all NWUE; these NBUE (NWUE) processing time distributions which all have mean one do not necessarily have to be the same. The following corollary follows immediately from Corollary 1 and Lemma 1.

COROLLARY 2. *If the processing times X''_{ij} have mean one and are* NBUE (NWUE), *then*

$$E(C''_{\max}) \leqq (\geqq) E(C^*_{\max})$$

and

$$E(\sum C_j'') \leqq (\geqq) E(\sum C_j^*).$$

A lower bound for $E(C''_{\max})$ and $E(C''_j)$ when the X_{ij} are arbitrarily distributed with mean one can be obtained by considering the case where all the processing times are deterministic and equal to one.

3. Flow shops and job shops: the independent case. In this section two machine flow shops with unlimited intermediate storage, two machine flow shops with no intermediate storage and two machine job shops are considered. The mn processing times are assumed to be independent and exponentially distributed with mean one. The results stated in the following theorem are mainly due to Milch and Waggoner (1972).

THEOREM 2.

(i) $$E(C^*_{\max}(F_{2,i,\infty})) = 2n\left[\left(\frac{1}{2}\right)^{2n}\sum_{i=n}^{2n}\binom{2n}{i} + 2^{-2n}\binom{2n-1}{n}\right].$$

(ii) $$E(C^*_{\max}(F_{2,i,0})) = 2 + (n-1)\frac{3}{2}.$$

(iii) $$E(C^*_{\max}(J_{2,i})) = 2n\left[\left(\frac{1}{2}\right)^{2n}\sum_{i=n}^{2n}\binom{2n}{i} + 2^{-2n}\left(\binom{2n-1}{n+p} + \binom{2n-1}{n+q}\right) - \frac{p}{n}\left(\frac{1}{2}\right)^{2n}\sum_{i=n+p+1}^{2n}\binom{2n}{i} - \frac{q}{n}\left(\frac{1}{2}\right)^{2n}\sum_{i=n+q+1}^{2n}\binom{2n}{i}\right].$$

Proof. (i) See Milch and Waggoner (1972).

(ii) When job $j, j=2, \cdots, n$, starts its processing on machine 1, job $j-1$ starts its processing on machine 2. The expected time that job $j, j=2, \cdots, n$ then occupies machine 1 is the maximum of two independent exponentially distributed random variables, which equals $\frac{3}{2}$. Moreover, the expected time that job 1 occupies machine 1 is 1 and the expected time that job n occupies machine 2 is 1 as well. The result then follows.

(iii) See Milch and Waggoner (1972).

By combining Theorem 2 with Corollary 2 bounds can be obtained for the expected makespan in case the processing time distributions G_{ij} have mean one and are NBUE (NWUE) and the processing times of a job on the two machines are independent. These processing time distributions do not necessarily have to be the same (there may be up to $2n$ different distributions). However, all of them must have mean one and be NBUE (NWUE). Closed form expressions can now also be obtained for $E(\sum C_j^*(F_{2,i,\infty}))$, $E(\sum C_j^*(F_{2,i,0}))$ and $E(\sum C_j^*(J_{2,i}))$. To obtain an expression for $E(\sum C_j^*(F_{2,i,\infty}))$, replace in the expression for $E(C^*_{\max}(F_{2,i,\infty}))$ the n by an l and sum over l from 1 to n.

4. Flow shops and job shops: the equal case. In this section first $C_{\max}(F_{m,e,\infty})$, $C_{\max}(F_{m,e,0})$ and $C_{\max}(J_{2,e})$ are determined as functions of the processing times $X_1, \cdots, X_n$ of the n jobs. Afterwards, closed form expressions are obtained for $E(C^*_{\max}(F_{m,e,\infty}))$, $E(C^*_{\max}(F_{m,e,0}))$ and $E(C^*_{\max}(J_{2,e}))$.

THEOREM 3.

(i) $$C_{\max}(F_{m,e,\infty}) = \sum_{j=1}^{n} X_j + (m-1)\max(X_1, \cdots, X_n).$$

$$\text{(ii)}\quad C_{\max}(F_{m,e,0}) = \sum_{j=1}^{m} \max(X_1, \cdots, X_j) + \sum_{j=m+1}^{n} \max(X_{j-m+1}, X_{j-m+2}, \cdots, X_j)$$

$$+ \sum_{i=2}^{m} \max(X_n, X_{n-1}, \cdots, X_{n-m+i}).$$

$$\text{(iii)}\quad C_{\max}(J_{2,e}) = \max\left(X_1, \cdots, X_p, \sum_{j=p+1}^{n} X_j\right) + \max\left(X_{p+1}, \cdots, X_n, \sum_{j=1}^{p} X_j\right).$$

Proof. (i) It suffices to show that

$$T_{ij} = \sum_{k=1}^{j} X_k + (i-1)\max(X_1, \cdots, X_j).$$

The proof is by induction. It is clearly true for $j=1, i=1, \cdots, m$ and for $i=1, j=1, \cdots, n$. Assume it is true for $j=l-1$ and $i=1, \cdots, m$. So

$$T_{i,l-1} =' \sum_{k=1}^{l-1} X_k + (i-1)\max(x_1, \cdots, X_{l-1})$$

$$= T_{1,l-1} + (i-1)\max(X_1, \cdots, X_{l-1}).$$

Two cases have to be considered, namely the case where $X_l < \max(X_1, \cdots, X_{l-1})$ and the case where $X_l > \max(X_1, \cdots, X_{l-1})$. When $X_l < \max(X_1, \cdots, X_{l-1})$, job l must wait for machine $i+1$, $i=1, \cdots, m-1$, after it has completed its processing on machine i, since job $l-1$ leaves machine $i+1$ a time period $\max(X_1, \cdots, X_{l-1})$ after job $l-1$ leaves machine i. So in this case

$$T_{i,l} = T_{i,l-1} + X_l.$$

When $X_l > \max(X_1, \cdots, X_{l-1})$, job l can immediately start its processing on machine $i+1$ after it has completed its processing on machine i. So in this case

$$T_{i,l} = T_{1,l} + (i-1)X_l.$$

Combining these two cases, it becomes clear that

$$T_{i,l} = T_{1,l} + (i-1)\max(X_1, \cdots, X_l).$$

(ii) First it is shown that the time job j, $j=1, \cdots, m$, spends on machine 1 is $\max(X_j, X_{j-1}, \cdots, X_1)$. Let job l be the job with the longest processing time among jobs $1, \cdots, j$, i.e.

$$X_l = \max(X_j, X_{j-1}, \cdots, X_1).$$

Because of the zero waiting room between the machines, it is clear that when job l starts on machine 2, job $l+1$ starts on machine 1; when job l starts on machine 3, job $l+1$ starts on machine 2, as $X_{l+1} < X_l$, and job $l+2$ starts on machine 1. Consequently, jobs $j, j-1, \cdots, l+1$ always move when job l does. Furthermore, jobs $l-1, l-2, \cdots, 1$ will not impede job l since their processing times are less than X_l. So the time job j, $j=1, \cdots, m$, spends on machine 1 is $\max(X_j, X_{j-1}, \cdots, X_1)$. In the same way it can be shown that the time job j, $j=m+1, \cdots, n$ spends on machine 1 is $\max(X_j, X_{j-1}, \cdots, X_{j-m+1})$ and that the time job n spends on machine i, $i=2, \cdots, m$, is $\max(X_n, X_{n-1}, \cdots, X_{n-m+i})$. The result follows.

(iii) Note that if neither one of the two machines ever remains idle in between the processing of jobs, the two machines finish with their processing exactly at the same time, as the amount of processing to be done, $\sum_{j=1}^{n} X_j$, is the same. If

$$\sum_{j=p+1}^{n} X_j \geqq \max(X_1, \cdots, X_p)$$

no idle periods occur on machine 2, and if

$$\sum_{j=1}^{p} X_j \geqq \max(X_{p+1}, \cdots, X_n)$$

no idle periods occur on machine 1. This implies that idle periods can only occur on one machine. Consider the case where

$$\sum_{j=p+1}^{n} X_j \leqq \max(X_1, \cdots, X_p).$$

In this case the two machine job shop can be regarded as a two machine flow shop with unlimited intermediate storage and $p+2$ jobs: one job, say job 0, has to go first and has zero processing time on machine 1 and a processing time of $\sum_{j=p+1}^{n} X_j$ on machine 2; jobs $1, \cdots, p$ follow job 0, going first through machine 1 with processing times $X_1, \cdots, X_p$ and afterwards through machine 2 again with processing times $X_1, \cdots, X_p$; one job, say job $n+1$, goes last, with a processing time of $\sum_{j=p+1}^{n} X_j$ on machine 1 and zero processing time on machine 2. An argument similar to (i) shows that the makespan then equals

$$\sum_{j=1}^{n} X_j + \max\left(X_1, \cdots, X_p, \sum_{j=p+1}^{n} X_j\right) - \sum_{j=p+1}^{n} X_j.$$

In this expression the third term is subtracted because job $n+1$ has zero processing time on machine 2. The total time machine 2 remains idle in between the processing of jobs is

$$\max\left(X_1, \cdots, X_p, \sum_{j=p+1}^{n} X_j\right) - \sum_{j=p+1}^{n} X_j$$

which is positive under the condition stated before. In a similar way, it can be shown that the total time machine 1 remains idle is

$$\max\left(X_{p+1}, \cdots, X_n, \sum_{j=1}^{p} X_j\right) - \sum_{j=1}^{p} X_j.$$

The makespan is the sum of $\sum_{j=1}^{n} X_j$, the idle time on machine 1 and the idle time on machine 2. At least one of the last two terms in this summation is zero. The result follows.

$E(C_{\max}(F_{m,e,\infty}))$ and $E(C_{\max}(F_{m,e,0}))$ can now be computed for various processing time distributions, e.g. the Uniform, the Exponential. To compute $E(C_{\max}(J_{2,e}))$ for the various processing time distributions is slightly more complicated.

THEOREM 4.

(i) $E(C^*_{\max}(F_{m,e,\infty})) = n + (m-1) \sum_{j=1}^{n} \frac{1}{j}.$

(ii) $E(C^*_{\max}(F_{m,e,0})) = 2 \sum_{j=1}^{m-1} \sum_{k=1}^{j} \frac{1}{k} + \sum_{j=m}^{n} \sum_{k=1}^{m} \frac{1}{k}.$

(iii) $E(C^*_{\max}(J_{2,e})) = n + \sum_{i=1}^{p} A_{i,p} + \sum_{i=1}^{n-p} A_{i,n-p},$

where

$$A_{i,p}=\frac{1}{i}\left\{1-(-1)^{i+1}\binom{p}{i}[1-(1/(i+1))^{n-p}]\right\}.$$

Proof. Parts (i) and (ii) follow immediately from parts (i) and (ii) of Theorem 3. Part (iii) is more complicated. Note that $\max(X_1,\cdots,X_p,\sum_{j=p+1}^n X_j)$, where $X_1,\cdots,X_n$ are independent exponentially distributed with mean one, has the same distribution as $\max(\sum_{j=1}^p Y_j,\sum_{j=p+1}^n X_j)$, where Y_j is exponentially distributed with mean $1/(p+1-j)$. Observe that

$$E\left(\max\left(\sum_{j=1}^p Y_j,\sum_{j=p+1}^n X_j\right)\right)=\sum_{j=1}^p\frac{1}{j}+E\left(\max\left(0,\sum_{j=p+1}^n X_j-\sum_{j=1}^p Y_j\right)\right).$$

Now

$$E\left(\max\left(0,\sum_{j=p+1}^n X_j-\sum_{j=1}^p Y_j\right)\right)=\sum_{j=0}^{n-p}(n-p-j)P\left(\sum_{i=p+1}^{p+j}X_i\leqq\sum_{i=1}^p Y_i\leqq\sum_{i=p+1}^{p+j+1}X_i\right).$$

Let $f(t)$ denote the probability density function of $\sum_1^p Y_i$. By induction, using the identity

$$\sum_{i=1}^{p-1}(-1)^{i+1}\binom{p}{i}i=(-1)^p p,$$

it can be shown that

$$f(t)=\sum_{i=1}^p(-1)^{i+1}\binom{p}{i}i\,e^{-it},\qquad t\geqq 0.$$

So

$$P\left(\sum_{p+1}^{p+j}X_i\leqq\sum_1^p Y_i<\sum_{p+1}^{p+j+1}X_i\right)=\int_0^\infty f(t)\frac{t^j e^{-t}}{j!}\,dt=\sum_{i=1}^p i(-1)^{i+1}\binom{p}{i}/(i+1)^{j+1}.$$

After some straightforward, but tedious, manipulations it follows that

$$\begin{aligned}\sum_{j=0}^{n-p}(n-p-j)P&\left(\sum_{p+1}^{p+j}X_i\leqq\sum_1^p Y_i<\sum_{p+1}^{p+j+1}X_i\right)\\&=\sum_{i=1}^p i(-1)^{i+1}\binom{p}{i}\sum_{j=0}^{n-p}(n-p-j)/(i+1)^{j+1}\\&=\sum_{i=1}^p i(-1)^{i+1}\binom{p}{i}\left[\frac{n-p}{i}-\left(\frac{1}{i}\right)^2\left(1-\left(\frac{1}{i+1}\right)^{n-p}\right)\right]\\&=(n-p)-\sum_{i=1}^p\frac{1}{i}(-1)^{i+1}\binom{p}{i}\left(1-\left(\frac{1}{i+1}\right)^{n-p}\right).\end{aligned}$$

Thus

$$\begin{aligned}E\left(\max\left(\sum_{j=1}^p Y_j,\sum_{p+1}^n X_j\right)\right)&=(n-p)+\sum_{i=1}^p\frac{1}{i}\left\{1-(-1)^{i+1}\binom{p}{i}\left[1-\left(\frac{1}{i+1}\right)^{n-p}\right]\right\}\\&=(n-p)+\sum_{i=1}^p A_{i,p}.\end{aligned}$$

This completes the proof of the theorem.

Just as in § 3, bounds can now be obtained for the expected makespan in case the processing times of job j, $j=1,\cdots,n$, on the n machines are identical and distributed according to a distribution G_j which has mean one and is NBUE (NWUE).

5. Open shops: The equal and the independent case. Pinedo and Ross (1982) considered the two machine open shop with the processing times of any given job on the two machines being independent and exponentially distributed with mean one. Furthermore, they assumed that jobs which have not yet received processing on either one of the two machines have priority over jobs which already have received processing on one of the two machines.

THEOREM 5.

(i) $$E(C^*_{\max}(0_{2,i}))=2n-\sum_{k=n}^{2n-1} n\binom{k}{n}\left(\frac{1}{2}\right)^k+\left(\frac{1}{2}\right)^n.$$

(ii) $$E(C^*_{\max}(0_{2,e}))=n+\left(\frac{1}{2}\right)^{n-1}+\sum_{j=2}^{n}\left(\frac{1}{2}\right)^{n-j+1}\left(\frac{2}{3}\right)^{j-2}.$$

Proof. For part (i) see Pinedo and Ross (1982).

(ii) It is clear that

$$C_{\max}(0_{2,e})\geqq\sum_{j=1}^{n}X_j.$$

If both machines work without interruption and no idle periods occur in between the processing of jobs on either one of the machines, then the makespan is equal to the R.H.S. in the above expression. However, an idle period can occur on one of the machines, say machine 1, when this machine has completed the processing of $n-1$ jobs and has to wait for the last job, say job j, because this last job is just then being processed on machine 2. After job j completes its processing on machine 2, leaving machine 1 idle for a time T which is exponentially distributed with mean one, it switches over to machine 1. Machine 1 then processes job j and completes this processing a time T after machine 2 completes all its processing. What remains to be computed is the probability that such an idle period occurs. When job j, $j>2$, starts its first processing $j-2$ jobs have already completed their first processing. Assume job j starts its first processing on machine 1. For job j to cause an idle period, it has to outlast $n-j+1$ jobs which have to complete their first processing on machine 2. It also has to outlast the second processing of those jobs that have completed their first on machine 1 before job j started its first processing; this amount of time, which is equal to the amount of time between $t=0$ and the time when job j initiates its first processing, is distributed according to a convolution of $j-2$ exponential distributions, each one having rate 2. So the probability that job j causes an idle period is

$$p_j=(\tfrac{1}{2})^{n-j+1}(\tfrac{2}{3})^{j-2},\qquad j=3,\cdots,n.$$

It can be shown easily that

$$p_1=p_2=(\tfrac{1}{2})^{n-1}.$$

Therefore

$$E(C^*_{\max}(0_{2,e}))=n+2\left(\frac{1}{2}\right)^{n-1}+\sum_{j=3}^{n}\left(\frac{1}{2}\right)^{n-j+1}\left(\frac{2}{3}\right)^{j-2}$$

and the result follows.

Unlike the situation in §§ 3 and 4, Theorem 1 cannot be used to establish bounds for the expected makespan when the processing times are NBUE (NWUE). In order to obtain bounds the effect of the idle period on the expected makespan has to be investigated. The following corollary follows immediately from the results of Pinedo and Ross and from Theorem 5.

COROLLARY 3. (i) *If the processing times in* $0_{2,i}$ *are* NBUE *and have mean one, then*

$$n \leqq E(C_{\max}(0_{2,i})) < 2n - \sum_{k=n}^{2n-1} n\binom{k}{n}\left(\frac{1}{2}\right)^k + 1.$$

If the processing times are NWUE *and have mean one, then*

$$2n - \sum_{k=n}^{2n-1} n\binom{k}{n}\left(\frac{1}{2}\right)^k < E(C_{\max}(0_{2,i})).$$

(ii) *If the processing times in* $0_{2,e}$ *are* NBUE *and have mean one, then*

$$n \leqq E(C_{\max}(0_{2,e})) < n+1.$$

If the processing times are NWUE *and have mean one, then*

$$n < E(C_{\max}(0_{2,e})).$$

6. Additional remarks. (i) Lemma 3 and Theorem 1 can be generalized easily in the following way. Consider a flow shop with no storage space in between the machines and let

$$X_{ij} = f_{ij}(Y_1, \cdots, Y_k),$$

where $Y_1, \cdots, Y_k$ are independent random variables and f_{ij} is a function which is increasing convex in $Y_1, \cdots, Y_k$. Now, making the random variables $Y_1, \cdots, Y_k$ more variable causes the random variables X_{ij} to become more variable and therefore also the makespan and the flow time. Through an appropriate choice of functions f_{ij} a flow shop with finite (and positive) intermediate storage between the machines can be constructed. This indicates that Theorem 1 does not only hold for flow shops with no intermediate storage and flow shops with infinite intermediate storage; it also holds for flow shops with finite and positive intermediate storage.

(ii) Lemma 3 and Theorem 1 also hold for m machine job shops where for each job the machine sequence is determined in advance and where for each machine the job sequence is determined in advance as well.

(iii) Lemma 3 and Theorem 1 fail to hold when the following situation is allowed to occur in a shop: A machine is allowed to start processing any one of a set of jobs dependent upon which one arrives first at this machine. The reason why Lemma 3 and Theorem 1 do not hold then lies in the fact that the minimum of a set of random arrival times has to be determined. This minimum is not an increasing convex function. This is also the reason why Lemma 2 and Theorem 1 do not hold for open shops.

(iv) The results in this paper indicate that a greater variability increases the expected makespan as well as the expected flow time. This is, however, not always the case in stochastic scheduling. Pinedo (1984) considered the case of m machines in *parallel* and n jobs. Each job has to be processed on one of the m machines; any one can do. It turns out that a greater variability indeed increases the expected makespan, but on the other hand, *decreases* the expected flow time.

REFERENCES

S. A. BESSLER AND A. F. VEINOTT, JR. (1966), *Optimal policy for a dynamic multi-echelon inventory model*, Naval Res. Logist. Quart., 13, pp. 355–389.

A. W. MARSHALL AND F. PROSHAN (1972), *Classes of distributions applicable in replacement, with renewal theory applications*, in Proc. 6th Berkeley Symposium on Mathematical Statistics and Probability, Vol. I, Univ. California Press, Berkeley, pp. 395–415.

P. R. MILCH AND M. H. WAGGONER (1972), *A random walk approach to a shutdown queueing system*, this Journal, 19, pp. 103–115.

S. C. NIU (1981), *On the comparison of waiting times in tandem queues*, J. Appl. Prob., 18, pp. 707–714.

M. L. PINEDO (1983), *Inequalities and bounds for the scheduling of stochastic jobs on parallel machines*, J. Appl. Prob., to appear.

——— (1984), *On optimal policies in stochastic shop scheduling*, Ann. Oper. Res. Vol. I, P. Hammer, ed., J. C. Baltzer Scientific Publishing Co., Basel, Switzerland.

M. L. PINEDO AND S. M. ROSS (1982), *Minimizing the expected makespan in stochastic open shops*, Adv. Appl. Prob., 14, pp. 898–911.

M. L. PINEDO AND L. SCHRAGE (1982), *Stochastic shop scheduling: a survey*, in Deterministic and Stochastic Scheduling, M. A. H. Dempster et al., eds., Reidel, Dordrecht, the Netherlands.

H. STOYAN AND D. STOYAN (1969), *Monotonicity properties of waiting times in the $GI/G/1$ model*, Z. Angew. Math. Mech., 49, pp. 729–734. (In German.)

W. WHITT (1980), *The effect of variability in the $GI/G/S$ queue*, J. Appl. Prob., 17, pp. 1062–1071.