MATHEMATICAL TRIPOS: PART IA **Lent 2015**

PROBABILITY **RRW**

Example Sheet 2 (of 4)

Exercises

1. A coin with probability p of heads is tossed n times. Let E be the event 'a head is obtained on the first toss' and F_k the event 'exactly k heads are obtained'. For which pairs of integers (n, k) are E and F_k independent?

2. The events A and B are independent. Show that the events A^C and B are independent, and that the events A^C and B^C are independent.

3. Independent trials are performed, each with probability p of success. Let P_n be the probability that n trials result in an even number of successes. Show that

$$P_n = \tfrac{1}{2}[1 + (1 - 2p)^n].$$

4. Two darts players A and B throw alternately at a board and the first to score a bull wins the contest. The outcomes of different throws are independent and on each of their throws A has probability p_A and B has probability p_B of scoring a bull. If A has first throw, calculate the probability of A winning the contest.

5. Suppose that X and Y are independent Poisson random variables with parameters λ and μ respectively. Find the distribution of $X + Y$. Prove that the conditional distribution of X, given that $X + Y = n$, is binomial with parameters n and $\lambda/(\lambda + \mu)$.

6. (i) The number of misprints on a page has a Poisson distribution with parameter λ, and the numbers on different pages are independent. What is the probability that the second misprint will occur on page r?

(ii) A proofreader studies a single page looking for misprints. She catches each misprint (independently of others) with probability $1/2$. Let X be the number of misprints she catches. Find $P(X = k)$. Given that she has found $X = 10$ misprints, what is the distribution of Y, the number of misprints she has not caught? How useful is X in predicting Y?

7. $X_1, \dots, X_n$ are independent, identically distributed random variables with mean μ and variance σ^2. Find the mean of

$$S^2 = \sum_{i=1}^{n} (X_i - \bar{X})^2, \quad \text{where } \bar{X} = \frac{1}{n} \sum_{1}^{n} X_i.$$

8. In a sequence of n independent trials the probability of a success at the ith trial is p_i. Show that mean and variance of the total number of successes are $n\bar{p}$ and $n\bar{p}(1 - \bar{p}) - \sum_i (p_i - \bar{p})^2$ where $\bar{p} = \sum_i p_i/n$. Notice that for a given mean, the variance is greatest when all p_i are equal.

9. Let $(X, Y) = (\cos\theta, \sin\theta)$ where $\theta = \frac{k\pi}{4}$ and k is a random variable such that $P\{k = r\} = 1/8$, $r = 0, 1, \dots, 7$. Show that $\mathrm{cov}(X, Y) = 0$, but that X and Y are not independent.

10. Let $a_1, a_2, \ldots, a_n$ be a ranking of the yearly rainfalls in Cambridge over the next n years: assume $a_1, a_2, \ldots, a_n$ is a random permutation of $1, 2, \ldots, n$. Say that k is a record year if $a_i > a_k$ for all $i < k$ (thus the first year is always a record year). Let $Y_i = 1$ if i is a record year and 0 otherwise. Find the distribution of Y_i and show that $Y_1, Y_2, \ldots, Y_n$ are independent. Calculate the mean and variance of the number of record years in the next n years.

11. Liam's bowl of spaghetti contains n strands. He selects two ends at random and joins them together. He repeats this until no ends are left. What is the expected number of spaghetti hoops in the bowl?

12. Sarah collects figures from cornflakes packets. Each packet contains one of n distinct figures. Each type of figure is equally likely. Show that the expected number of packets Sarah needs to buy to collect a complete set of n is

$$n \sum_{i=1}^{n} \frac{1}{i}.$$

[*After doing this, you might like to visit the Wikipedia article about the 'Coupon collector's problem'.*]

13. (X_k) is a sequence of independent identically distributed positive random variables where $E(X_k) = a$ and $E(X_k^{-1}) = b$ exist. Let $S_n = \sum_{k=1}^{n} X_k$. Show that $E(S_m/S_n) = m/n$ if $m \leq n$, and $E(S_m/S_n) = 1 + (m-n)aE(S_n^{-1})$ if $m \geq n$. [*This was a Cambridge entrance exam question c. 1970.*]

Problems

These next questions are more challenging. I hope you will learn and have fun by attempting them.

14. You wish to use a fair coin to simulate occurrence or not of an event A that happens with probability 1/3. One method is to start by tossing the coin twice. If you see HH say that A occurred, if you see HT or TH say that A has not occurred, and if you see TT then repeat the process. Show that this enables you to simulate the event using an expected number of tosses equal to 8/3.

Can you do better? (i.e. simulate something that happens with probability 1/3 using a fair coin and with a smaller expected number of tosses) Hint. The binary expansion of 1/3 is 0.0101010101 . . . .]

15. Let X be an integer-valued random variable with distribution

$$P(X = n) = n^{-s}/\zeta(s)$$

where $s > 1$, and $\zeta(s) = \sum_{n \geq 1} n^{-s}$. Let $p_1 < p_2 < p_3 < \cdots$ be the primes and let A_k be the event $\{X$ is divisible by $p_k\}$. Find $P(A_k)$ and show that the events $A_1, A_2, \ldots$ are independent. Deduce that

$$\prod_{k=1}^{\infty} (1 - p_k^{-s}) = 1/\zeta(s).$$

16. You are playing a match against an opponent in which at each point either you or your opponent serves. If you serve you win the point with probability p_1, but if your opponent serves you win the point with probability p_2. There are two possible conventions for serving:

(i) serves alternate;
(ii) the player serving continues to serve until she loses a point.

You serve first and the first player to reach n points wins the match. Show that your probability of winning the match does not depend on the serving convention adopted.

[*Hint:* Under either convention you serve at most n times and your opponent at most $n-1$ times. Recall Pascal and Fermat's 'problem of the points', treated in lectures.]

Puzzles

This section is for enthusiasts — or for discussion in supervision when you have done everything else. The following puzzles have been communicated to me by Peter Winkler.

17. Let $k < n$, k even, n odd. Joey is to play n chess games against his parents, alternating between his father and mother. To receive his allowance he must win k games in a row. Prove that given the choice, he should start against the stronger parent.

[Hint: start by solving the cases $k = 2, n = 3$, and $k = n - 1$.]

18. Let $X_1, \ldots, X_6$ be i.i.d. $B(1, p)$ with $p = 0.4$. Let $S_n = X_1 + \cdots + X_n$. Argue that

$$P(S_4 \geq 3) = P(S_6 \geq 4)$$

without explicitly computing either the left or right hand sides.

[Hint. Compare $P(S_4 \geq 3 \mid S_5 = i)$ and $P(S_6 \geq 4 \mid S_5 = i)$ for each of $i = 0, \ldots, 5$.]