MATHEMATICAL TRIPOS: PART IA **Lent 2015**

PROBABILITY **RRW**

Example Sheet 1 (of 4)

Each sheet contains about 12 fairly straightforward 'Exercises', a few more challenging and/or lengthy 'Problems', and also a couple 'Puzzles'. Please work through the Exercises and Problems. The Puzzles are for enthusiasts, and might be fun to talk about in supervision if you have done everything else.

Exercises

1. Four mice are chosen (without replacement) from a litter, two of which are white. The probability that both white mice are chosen is twice the probability that neither is chosen. How many mice are there in the litter?

2. A table-tennis championship for 2^n equally good players is organized as a knock-out tournament with n rounds, the last round being the final. Two players are chosen at random. Calculate probabilities they meet

(i) in the first round,
(ii) in the final,
(iii) in any round.

[Hint: Can the same sample space be used for all three calculations?]

3. A full deck of 52 cards is divided into half at random. Use Stirling's formula to estimate the probability that each half contains the same number of red and black cards.

4. You throw $6n$ dice at random. Show that the probability that each number appears exactly n times is

$$\frac{(6n)!}{(n!)^6}\left(\frac{1}{6}\right)^{6n}.$$

Use Stirling's formula to show that this is approximately $cn^{-5/2}$ for some constant c to be found.

5. The first axiom of probability was stated as "I. $0 \le P(A) \le 1$ for all $A \subseteq \Omega$". Show that equivalent axioms are ones in which this is changed to the weaker requirement: "I. $P(A) \ge 0$ for all $A \subseteq \Omega$".

6. (i) If A, B, C are three events, show that

$$P(A^c \cap (B \cup C)) = P(B) + P(C) - P(B \cap C) - P(C \cap A) - P(A \cap B) + P(A \cap B \cap C).$$

(ii) How many of the numbers $1, \dots, 500$ are not divisible by 7 but are divisible by 3 or 5?

7. A coin is tossed repeatedly. The outcomes of the tosses are independent. Let A_k be the event that the kth toss is a head, $k = 1, 2, \dots$.

(i) Explain in words what happens when the event $B = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k^c$ occurs.
(ii) Express, similarly as in (i), the event C, that 'an infinite number of heads occur'.
(iii) What do you think are the probabilities of B and C when $P(A_k) = p$ for all k and $0 < p < 1$? Can you prove this?

(iv) Use the properties of P as set out in Lecture 4 to give a rigorous calculation of $P(B)$ when the problem is changed so that tosses are independent, but are made with coins of different biases, such that $P(A_k) = p_k$ and $\sum_k p_k < \infty$.

8. Let $A_1, \dots, A_m \subseteq \{1, \dots, n\}$ be finite sets with $A_i \not\subset A_j$ and let $a_i = |A_i|$. Let $\sigma_1, \dots, \sigma_n$ be a randomly chosen permutation of $1, 2, \dots, n$ and let E_i be the event $\{\sigma_1, \dots, \sigma_{a_i}\} = A_i$. Are E_1 and E_2 disjoint? Are E_1 and E_2 independent?

Prove that

$$\sum_{i=1}^{m} \binom{n}{a_i}^{-1} \leq 1.$$

9. A committee of size r is chosen at random from a set of n people. Calculate the probability that m given people will all be on the committee (a) directly, and (b) using the inclusion-exclusion formula. Deduce that

$$\binom{n-m}{r-m} = \sum_{j=0}^{m} (-1)^j \binom{m}{j} \binom{n-j}{r}.$$

10. Let $A_1, \dots, A_n$ be events. Prove the following improvement of Boole's inequality.

$$P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i) - \sum_{i=1}^{n-1} P(A_i \cap A_{i+1}).$$

[Hint. Induction.] Deduce that

$$P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i) - \frac{2}{n} \sum_{1 \leq i_1 < i_2 \leq n} P(A_{i_1} \cap A_{i_2}).$$

11. Examination candidates are graded into four classes known conventionally as I, II–1, II–2 and III, with probabilities 1/8, 2/8, 3/8 and 2/8 respectively. A candidate who misreads the rubric, — a common event with probability 2/3 —, generally does worse, his or her probabilities being 1/10, 2/10, 4/10 and 3/10. What is the probability:

(i) that a candidate who reads the rubric correctly is placed in the class II–1?
(ii) that a candidate who is placed in the class II–1 has read the rubric correctly?

12. Parliament contains a proportion p of Labour members, who are incapable of changing their minds about anything, and a proportion $1 - p$ of Conservative members who change their minds completely at random (with probability r) between successive votes on the same issue. A randomly chosen member is noticed to have voted twice in succession in the same way. What is the probability that this member will vote in the same way next time?

Problems

Some of these are more challenging. I hope you will learn and have fun by attempting them.

13. Mary tosses two coins and John tosses one coin. What is the probability that Mary gets more heads than John? Answer the same question if Mary tosses three coins and John tosses two. Make a conjecture for the probability when Mary tosses $n + 1$ and John tosses n. Can you prove your conjecture?

14. [*This simply stated problem requires thought in modelling the sample space.*] Suppose that n balls are tossed independently and at random into n boxes. What is the probability that exactly one box is empty? Check your formula gives the correct answer for $n = 2$ and $n = 3$ (for which the probabilities are 1/2 and 2/3, respectively).

[Hint. In a problem like this it is up to you to decide whether you will imagine that the balls and/or boxes are (or are not) distinguishable. You may need to experiment until you find the model that best enables you to calculate the desired probability.]

15. The Polya urn model for contagion is as follows. We start with an urn which contains one white ball and one black ball. At each second we choose a ball at random from the urn and replace it together with one more ball of the same colour. Calculate the probability that when n balls are in the urn, i of them are white. (You might like to carry out a computer simulation — do you think the proportion of white balls might tend to a limit?)

16. Suppose a die is rolled n times. Show that the probability of a roll of i appearing k_i times, $i = 1, \dots, 6$, is

$$\phi(k_1, \dots, k_6) = \frac{n!}{k_1! \cdots k_6!} \frac{1}{6^n}.$$

The expected value of the total of n rolls is $3.5n$. Suppose $\sum_i ik_i = \rho n$, $1 \le \rho \le 6$, and $k_i = np_i$. Use Stirling's formula to show that subject to these constraints, ϕ is maximized by choosing the p_i as nonnegative numbers that solve the optimization problem

$$\underset{p_1,\dots,p_6}{\text{maximize}} - \textstyle\sum_i p_i \log p_i, \quad \text{subject to } \textstyle\sum_i p_i = 1, \text{ and } \textstyle\sum_i ip_i = \rho.$$

For $\rho = 4$ the maximizer is $p^* \approx (0.103, 0.123, 0.146, 0.174, 0.207, 0.247)$, so the most likely way of obtaining a total of $4n$ is when about 24.7% of the dice rolls are a 6. What do you guess would be the solution to the optimization problem if $\rho = 3.5$? If $\rho = 3$?

Puzzles

These are for enthusiasts, or to discuss in supervision when you have done everything else.

17. I am playing poker with three friends and from a well-shuffled deck we have each been dealt five cards. I have a hand consisting of the four kings and the two of hearts. Being a poker wizard I know exactly the probability that I have a winning hand.

But then I suddenly discover that earlier in the day the family's children were playing with the cards and fed six of them to their pet goat (but I don't know which). How does this information change the probability with which I believe my hand will beat my opponents? Why?

18. Suppose that there are 42 bags, labeled 0 though 41. Bag i contains i red balls and $41 - i$ blue balls. Suppose that you pick a bag at random, then pull out three balls without replacement. What is the probability that all 3 balls are the same colour?