Optimization - Examples Sheet 2

1. Consider the three equations in 6 unknowns given by $Ax = b$ where

$$A = \begin{pmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 2 & 2 & 1 & 0 & 0 & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} 2 \\ 5 \\ 6 \end{pmatrix}.$$

Choose B $= \{1, 3, 6\}$ and write $Ax = b$ in the form $A_B x_B + A_N x_N = b$ where $x_B = (x_1, x_3, x_6)^\top$, $x_N = (x_2, x_4, x_5)^\top$ and the matrices A_B and A_N are constructed appropriately.

Now, write $c^\top x = c_B^\top x_B + c_N^\top x_N$ and hence write $c^\top x$ in terms of the matrices A_B, A_N and the variables x_N (i.e., eliminate the variables x_B).

Compute A_B^{-1} and hence calculate the basic solution having B as basis. For $c = (3, 1, 3, 0, 0, 0)^\top$ write $c^\top x$ in terms of the non-basic variables. Prove directly from the formula for $c^\top x$ that the basic solution you have computed is optimal for the problem maximize $c^\top x$ subject to $x \geq 0$, $Ax = b$.

Compare your answer with your answer to question 8 on sheet 1 and confirm that the final tableau had rows corresponding to the equation $x_B + A_B^{-1} A_N x_N = A_B^{-1} b$. Why is it not fair to say that the simplex algorithm is just a complicated way to invert a matrix?

2. Take problem P from question 7 on sheet 1 and add the additional constraint $x_1 + 3x_2 \leq 6$. Apply the simplex algorithm putting x_2 into the basis at the first stage. Show that the solution at $x_1 = 0$, $x_2 = 2$ is degenerate. Explain, with a diagram, what happens.

3. In the previous question the additional constraint was redundant (it did not alter the feasible set). Can degeneracy occur without redundant equations?

4. Show that introducing slack variables in a LP does not change the extreme points of the feasible set by proving (using the definition of an extreme point) that x is an extreme point of $\{x : x \geq 0, Ax \leq b\}$ if and only if $\binom{x}{z}$ is an extreme point of $\{\binom{x}{z} : x \geq 0, z \geq 0, Ax + z = b\}$.

5. Give sufficient conditions for strategies p and q to be optimal for a two-person zero-sum game with pay-off matrix A and value v.

Two players fight a duel: they face each other $2n + 1$ paces apart and each has a single bullet in his gun. At a signal each may fire. If either is hit or if both fire the game ends. Otherwise both advance one pace and may again fire. The probability of either hitting his opponent if he fires after the i^{th} pace forward $(i = 0, 1, \ldots, n)$ is i/n. If a player survives after his opponent has been hit his payoff is $+1$ and his opponent's payoff is -1. The payoff is 0 if neither or both are hit. The guns are silent so that neither knows whether or not his opponent has fired.

Show that, if $n = 4$, the strategy 'shoot after taking two steps' is optimal for both, but that if $n = 5$ a mixed strategy is optimal. [*Hint*: $(0, \frac{5}{11}, \frac{5}{11}, 0, \frac{1}{11})$.]

6. By considering the payoff matrix

$$A = \begin{pmatrix} 0 & -2 & 3 & 0 \\ 2 & 0 & 0 & -3 \\ -3 & 0 & 0 & 4 \\ 0 & 3 & -4 & 0 \end{pmatrix}.$$

show that the optimal strategies for a two-person zero-sum game are not necessarily unique. Find all optimal strategies.

7. The $n \times n$ matrix of a two-person zero-sum game is such that the row and column sums all equal s. Show that the game has value s/n.

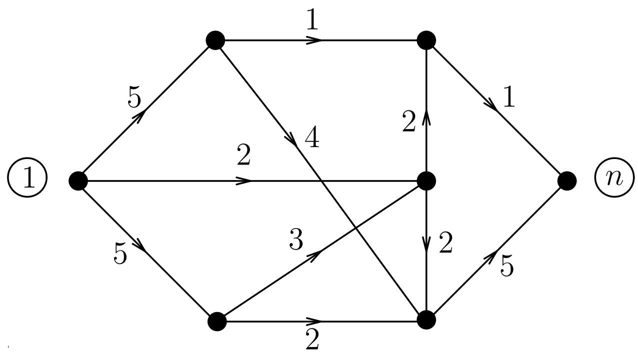
[Hint: Guess a solution and show that it is optimal.]

8. Find optimal strategies for both players, and the value of the game, for the game with payoff matrix

$$A = \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}.$$

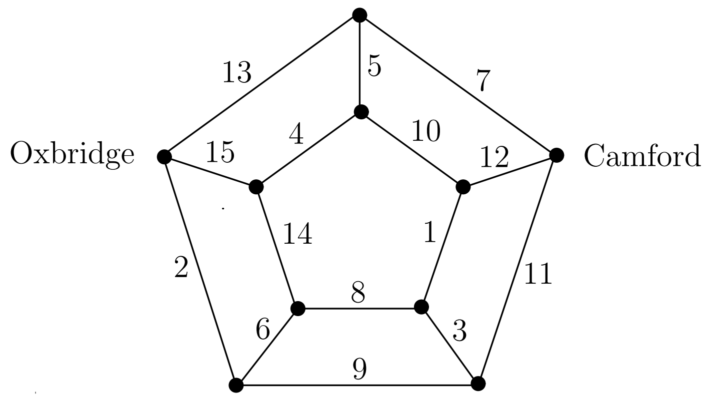
[You may like to try to compare the effort required to solve this by (a) seeking strategies and a value which satisfy the optimality conditions (b) direct solution of Player I's original minimax problem and (c) using the simplex method on one of the players problems after transforming it as suggested in the lectures.]

9. Find a 'maximal flow' and 'minimal cut' for the network below with a source at node 1 and sink at node n.



10. Devise rules for a version of the Ford-Fulkerson algorithm which works with undirected arcs.

As a consequence of drought, an emergency water supply must be pumped from Oxbridge to Camford along the network of pipes shown in the figure. The numbers against the pipes show their maximal capacities, and each pipe may be used in either direction. Find the maximal flow and prove that it is maximal.



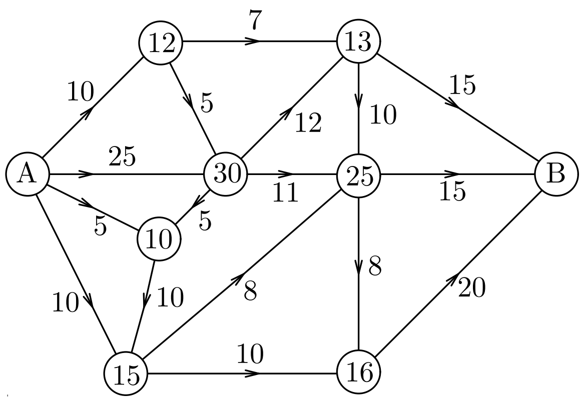
11. (Konig-Egervary Theorem). Consider an $m \times n$ matrix A in which every entry is either 0 or 1. Say that a set of lines (rows or columns of the matrix) *covers* the matrix if each 1 belongs to some line of the set. Say that a set of 1's is *independent* if no pair of 1's of the set lies in the same line. Use the max-flow min-cut theorem to show that the maximal number of independent 1's equals the minimal number of lines that cover the matrix.

12. Suppose we deal 52 cards into 13 piles of 4 cards each. Show that it is always possible to select exactly 1 card from each pile, such that the 13 selected cards contain exactly one card of each rank (A, 2, 3,..., Q, K).

[Hint: Consider a graph having 80 nodes, labeled $A, a_1, \ldots, a_{13}, b_1, \ldots, b_{52}, c_1, \ldots, c_{13}, B$. For each i $(= 1, \ldots, 13)$ let the graph have arcs of the form (A, a_i) and (c_i, B), each of capacity 1. For each i let there be exactly four more arcs going from a_i to some 4 members of $\{b_1, \ldots, b_{52}\}$, and exactly four arcs going from this set to each c_i, all of capacity ∞. Each b_j has only one arc incoming from some a_i, and one arc outgoing to some c_i. Show that in this network the min-cut (between A and B) is 13.]

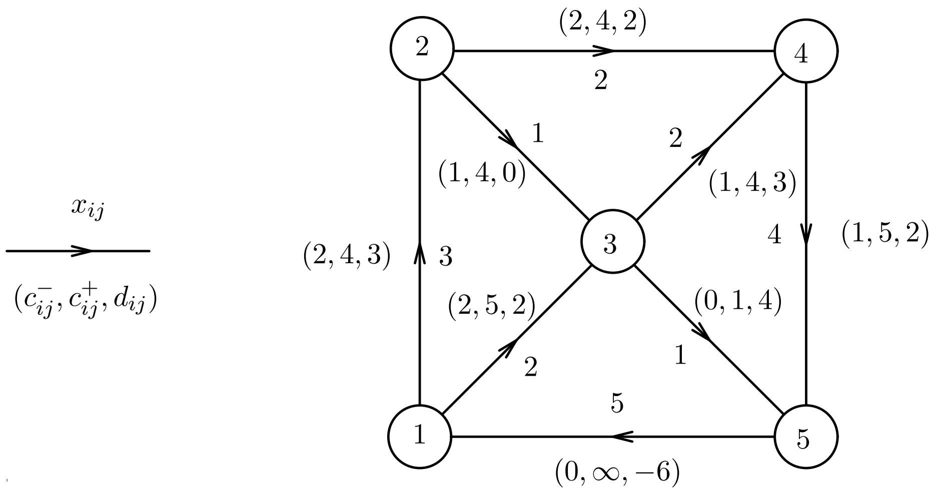
13. How would you augment a directed network to incorporate restrictions on node capacity (the total flow permitted through a node) in maximal flow problems?

The road network between two towns A and B is represented in the diagram below. Each road is marked with an arrow giving the direction of the flow, and a number which represents its capacity. Each of the nodes of the graph represents a village. The total flow into a village cannot exceed its capacity (the number in the circle at the node). Obtain the maximal flow from A to B.



The Minister of Transport intends to build a by-pass round one of the villages, whose effect would be to completely remove the capacity constraint for that village. Which village should receive the by-pass if her intent is to increase the maximal flow from A to B as much as possible? What would the new maximal flow be?

14. By finding a suitable potential on the nodes (i.e., a set of suitable node numbers) of the network show that the flow illustrated below is a minimal cost circulation. [Each arc is labelled with x_{ij} and with a triple of numbers giving the constants $(c_{ij}^-, c_{ij}^+, d_{ij})$ for that arc.]



15. Lecturers $L_1, \ldots, L_n$ are to be assigned to courses $C_1, \ldots, C_n$ so as to minimize student dissatisfaction. The dissatisfaction felt by students if lecturer L_i gives course C_j is d_{ij}, and each lecturer must give exactly one course. Show how to state this problem as the problem of minimizing the cost of a circulation in a network.

For the example of 3 lecturers and 3 courses with dissatisfaction matrix

$$\begin{pmatrix} 6 & 3 & 1 \\ 8 & 12 & 5 \\ 3 & 11 & 7 \end{pmatrix}$$

find an optimal flow through the appropriate network (by guessing) and compute node numbers for each node so that the optimality conditions are satisfied.

16. Sources 1, 2, 3 stock candy floss in amounts of 20, 42, 18 tons respectively. The demands for candy-floss at destinations 1, 2, 3 are 39, 34, 7 tons respectively. The matrix of transport costs per ton is

$$\begin{pmatrix} 7 & 4 & 9 \\ 8 & 12 & 5 \\ 3 & 11 & 7 \end{pmatrix}$$

with the (ij) entry corresponding to the route $i \to j$. Find the optimal transportation scheme and the associated minimal cost.