Optimization - Examples Sheet 1

1. Show how to solve the problem

$$\min \sum_{i=1}^{n} \frac{1}{(a_i + x_i)} \qquad \text{subject to} \qquad \sum_{i=1}^{n} x_i = b, \quad x_i \geq 0 \quad (i = 1, \dots, n)$$

where $a_i > 0$, $i = 1, \dots, n$ and $b > 0$.

2. Minimize each of the following functions in the region specified.

(a) $3x$ in $\{x : x \geq 0\}$; (b) $x^2 - 2x + 3$ in $\{x : x \geq 0\}$; (c) $x^2 + 2x + 3$ in $\{x : x \geq 0\}$.

For each of the following functions specify the set Y of λ values for which the function has a finite minimum in the region specified, and for each $\lambda \in Y$ find the minimum value and (all) optimal x.

(d) λx subject to $x \geq 0$; (e) λx subject to $x \in \mathbb{R}$; (f) $\lambda_1 x^2 + \lambda_2 x$ subject to $x \in \mathbb{R}$;

(g) $\lambda_1 x^2 + \lambda_2 x$ subject to $x \geq 0$; (h) $(\lambda_1 - \lambda_2)x$ subject to $0 \leq x \leq M$.

3. Maximize $n_1 \log p_1 + \ldots + n_k \log p_k$ subject to $p_1 + \ldots + p_k = 1$, $p_1, \dots, p_k > 0$, where $n_1, \dots, n_k$ are given positive constants. [The optimal $(p_1, \dots, p_k)$ is the maximum likelihood estimator for the multinomial distribution, $p(n_1, \dots, n_k) = \binom{n}{n_1, \dots, n_k} p_1^{n_1} \cdots p_k^{n_k}$.]

4. A *probability vector* is a vector $p = (p_1, \dots, p_n)^\top$ with $p_i \geq 0$ for all i and $\sum_{i=1}^{n} p_i = 1$. The entropy $H(p)$ is defined by

$$H(p) = -\sum_{i=1}^{n} p_i \log p_i,$$

where $0 \log 0 = 0$ by convention. Find the maximum and the minimum values of $H(p)$ over probability vectors p.

5. Maximize $2 \tan^{-1} x_1 + x_2$ subject to $x_1 + x_2 \leq b_1$, $-\log x_2 \leq b_2$, $x_1 \geq 0$, $x_2 \geq 0$, where b_1, b_2 are constants such that $b_1 - e^{-b_2} \geq 0$.

[Hint. Think carefully about two cases in which the Lagrange multiplier for the second constraint is either $= 0$ or > 0.]

6. Write down the Lagrangian for each of the following problems. In each case find the set Y of λ values for which the Lagrangian has a finite minimum (subject to the appropriate regional constraint), calculate the minimum $L(\lambda)$ for each $\lambda \in Y$, and write down the dual problem. In each case, write down the conditions for primal and dual feasibility and any additional conditions (the complementary slackness conditions) needed for optimality.

(a) minimize $c^\top x$ subject to $Ax \leq b$, $x \geq 0$; (b) minimize $c^\top x$ subject to $Ax = b$, $x \geq 0$.

7. Let P be the linear problem

$$\begin{array}{lrrl} \text{maximize} & x_1 & + \ x_2 & \\ \text{subject to} & 2x_1 & + \ x_2 & \leq \ 4 \\ & x_1 & +2x_2 & \leq \ 4 \\ & x_1 & - \ x_2 & \leq \ 1 \\ & & x_1, x_2 & \geq \ 0. \end{array}$$

(a) Solve P graphically in the x_1, x_2 plane.

(b) Introduce slack variables x_3, x_4, x_5 and extend c and A to rewrite P as

$$\max \ c_e^\top x_e \quad \text{subject to } A_e x_e = b, \ x_e \geq 0.$$

Determine the variables $x_1, \ldots, x_5$ and of the objective function for all basic solutions to the equation $A_e x = b$.

Which of the basic solutions are the basic feasible solutions? Are all the basic solutions non-degenerate?

(c) Find the problem D that is dual to P. Introduce slack variables λ_4 and λ_5 into this problem and write down the value of the variables $\lambda_1, \ldots \lambda_5$ and of the objective function for each basic solution of D. Which are the basic feasible solutions of D?

(d) Let $z_1 = x_3$, $z_2 = x_4$, $z_3 = x_5$ be the slack variables for P and $v_1 = \lambda_4$, $v_2 = \lambda_5$ be the slack variables for D. Show that for each basic solution x of P there is exactly one basic solution λ of D such that (i) $c_e^\top x_e = b^\top \lambda$ (same objective value) and (ii) $\lambda_i z_i = 0$, $i = 1, 2, 3$ and $x_j v_j = 0$, $j = 1, 2$ (complementary slackness).

For how many pairs $\{x_e, \lambda\}$ is x_e feasible for P and λ feasible for D?

(e) Solve problem P using the simplex algorithm starting with initial basic feasible solution $x_1 = x_2 = 0$. Try both choices of the variable to put into the basis on the first step. Compare the objective rows of the various tableaux generated with appropriate basic solutions to problem D? What do you observe?

8. Use the simplex algorithm to solve

$$\begin{array}{lrl} \text{maximize} & 3x_1 + \ x_2 + 3x_3 & \\ \text{subject to} & 2x_1 + \ x_2 + \ x_3 & \leq \ 2 \\ & x_1 + 2x_2 + 3x_3 & \leq \ 5 \\ & 2x_1 + 2x_2 + \ x_3 & \leq \ 6 \\ & x_1, x_2, x_3 & \geq \ 0. \end{array}$$

Each row of the final tableau is the sum of scalar multiples of the rows of the initial tableau. Explain how to determine the scalar multipliers directly from the final tableau.

Let P(ϵ) to be the LP problem obtained by replacing the vector $b = (2, 5, 6)^\top$ by the perturbed vector $b(\epsilon) = (2 + \epsilon_1, 5 + \epsilon_2, 6 + \epsilon_3)^\top$. Give a formula, in terms of $\epsilon = (\epsilon_1, \epsilon_2, \epsilon_3)$, for the optimal value for P(ϵ) when the ϵ_i are small. If $\epsilon_2 = \epsilon_3 = 0$, for what range of ϵ_1 values does the formula hold?

9. Apply the simplex algorithm to

$$\begin{array}{lrcl} \text{maximize} & x_1 + 3x_2 & & \\ \text{subject to} & x_1 - 2x_2 & \leq & 4 \\ & -x_1 + \ x_2 & \leq & 3 \\ & x_1, x_2 & \geq & 0. \end{array}$$

Explain what happens with the use of a diagram.

10. Use the two-phase algorithm to solve:

$$\begin{array}{lrcl} \text{maximize} & -2x_1 - 2x_2 & & \\ \text{subject to} & 2x_1 - 2x_2 & \leq & 1 \\ & 5x_1 + 3x_2 & \geq & 3 \\ & x_1, x_2 & \geq & 0. \end{array}$$

Hint: You should get $x_1 = \frac{9}{16}$, $x_2 = \frac{1}{16}$. Note that it is possible to choose the first pivot column so that phase I last only one step. But this requires a different choice of pivot column than the one specified by the usual rule-of-thumb.

11. Use the two-phase algorithm to solve:

$$\begin{array}{lrcr} \text{minimize} & 13x_1 + 5x_2 - 12x_3 & & \\ \text{subject to} & 2x_1 + \ x_2 + \ 2x_3 & \leq & 5 \\ & 3x_1 + 3x_2 + \ \ x_3 & \geq & 7 \\ & x_1 + 5x_2 + \ 4x_3 & = & 10 \\ & x_1, x_2, x_3 & \geq & 0. \end{array}$$

12. Consider the problem

$$\begin{array}{lrcl} \text{minimize} & 2x_1+3x_2+5x_3+2x_4+3x_5 & & \\ \text{subject to} & x_1+ \ x_2+2x_3+ \ x_4+3x_5 & \geq & 4 \\ & 2x_1-2x_2+3x_3+ \ x_4+ \ x_5 & \geq & 3 \\ & x_1,\ x_2,\ x_3,\ x_4,\ x_5 & \geq & 0. \end{array}$$

Write down the dual problem, and solve this graphically. Hence deduce the optimal solution to the primal problem.