Search for a Moving Target

An object moves back and forth between two locations according to a discrete-time Markov chain with transition matrix

$$\begin{pmatrix} \bar{p}_1 & p_1 \\ p_2 & \bar{p}_2 \end{pmatrix}$$

- To search location i costs c_i.
- If we search location i and the object is there it is found with probability α_i.
- The aim is to minimize the expected cost of finding the object.

Given that at time 0 the object is in locations 1 or 2 with probabilities y and $\bar{y}$ respectively, we have

$$F(y) = \min \begin{cases} c_1 + (y\bar{\alpha}_1 + \bar{y}) F\left(\frac{y\bar{\alpha}_1\bar{p}_1 + \bar{y}p_2}{y\bar{\alpha}_1 + \bar{y}}\right) \\ c_2 + (y + \bar{y}\bar{\alpha}_2) F\left(\frac{y\bar{p}_1 + \bar{y}\bar{\alpha}_2 p_2}{y + \bar{y}\bar{\alpha}_2}\right) \end{cases}$$

where, for example, $\frac{y\bar{\alpha}_1\bar{p}_1 + \bar{y}p_2}{y\bar{\alpha}_1 + \bar{y}}$ is the probability that the object is now in location 1 given that we searched location 1 and did not find it.

This represents the problem in *closed-loop* form.

However, we could take all our decisions about where to search right at the start (in *open-loop* fashion), since as long as we have not yet found the object there is no helpful information to be gained.

Suppose that on the tth step we search locations 1 and 2 with probabilities u_t and $\bar{u}_t$ respectively. Let x_t and $\bar{x}_t$ be the probabilities that at time t the object has not yet been found and is in location 1 or 2 respectively. The dynamics are

$$\begin{aligned} x_{t+1} &= u_t(x_t\bar{\alpha}_1\bar{p}_1 + \bar{x}_t p_2) + \bar{u}_t(x_t\bar{p}_1 + \bar{x}_t\bar{\alpha}_2\bar{p}_2) \\ \bar{x}_{t+1} &= u_t(x_t\bar{\alpha}_1 p_1 + \bar{x}_t\bar{p}_2) + \bar{u}_t(x_t p_1 + \bar{x}_t\bar{\alpha}_2 p_2) \end{aligned}$$

with $x_0 = y_0$, $\bar{x}_0 = \bar{y}_0$, and we want to minimize

$$\sum_{t=0}^{\infty} (x_t + \bar{x}_t)(u_t c_1 + \bar{u}_t c_2).$$

Conjecture. *There exists a number θ (depending on p_1, p_2, α_1, α_2, c_1, c_2) such that an optimal policy is to search location 1 if and only if $x_t/\bar{x}_t \geq \theta$.*

Despite the intuitive plausibility of this conjecture, it has remained a difficult unsolved problem for many years.

A proof in continuous-time (when the object moves according to a continuous-time Markov process) was given in 1986 (Weber).