Part III. Continuous Time Models

Lectures 12–16 have covered continuous time deterministic dynamic programming and the Pontryagin maximum principle (PMP). Controlled Markov jump processes and diffusion processes are examples of controlled stochastic models in continuous time. Good references for this part of the course are Whittle, P., *Optimization over Time, volume I*, Chapter 7, *volume II*, Chapter 38, and for the PMP, Hocking, L.M., *Optimal Control: An introduction to the theory and applications*, Oxford 1991, chapters 4–7.

As a result of studying this material you should be able to

- explain the meaning of the key terms used in lectures and listed overleaf.
- make use of the standard notation summarised overleaf.
- recall illustrative examples of various problem types (e.g., harvesting, skateboard, Bush problem, queueing control, passage to a stopping set, etc.)
- know the following results and conditions under which they hold:
  - Theorem 12.1 form of the optimality equation in continuous time and its sufficiency in establishing the optimality of a policy from its value function.
  - Theorem 13.1 statement of the Pontryagin maximum principle.
  - Theorem 13.2 statement of the transversality conditions for the PMP.
  - Section 13.4 heuristic interpretation of adjoint variables and transversality conditions via a Lagrangian multiplier apporach.
  - Section 14.1 the PMP with time-dependent plant equation and/or costs.
  - Section 15.1 optimality equation in terms of infinitesimal generator.
  - Section 15.3 uniformized form of optimality equation for Markov jump process.
  - Section 16.1 optimality equation for diffusion process.
- solve problems that are similar to those in lectures and on Examples Sheet 3, by use of the continuous time optimality equation, PMP, and dynamic programming equations for Markov jump processes and diffusions.
- derive optimality equations for continuous time models via a limiting argument and consideration of the dynamic programming equation over time interval $[t, t + \delta t)$.

Key terms in Lectures 12–16

Notation in Lectures 12–16

x, u, t	state, control and time
$\dot{x} = a(x, u, t)$	continuous time form of the plant equation
$\mathcal{S}$	stopping set
T	time horizon, i.e., time t at which $(x(t), t)$ first enters $\mathcal{S}$
$c(x, u, t)$	cost at time t
$K(x(T), T)$	terminal cost
α	parameter in discounting, i.e., $C = \int_{t=0}^{T} c(x, u, t)e^{-\alpha t}\, dt + e^{-\alpha T} K(x(T), T)$
λ	adjoint variables, i.e., $\lambda_i = -\partial F/\partial x_i$, $i = 1, \dots, n$, (or $\lambda = -F_x$)
$H(x, u, \lambda)$	Hamiltonian, i.e., $H = \lambda^\top a - c$
λ_0	adjoint variable for time, i.e., $\lambda_0 = -\partial F/\partial t$ (or $\lambda_0 = -F_t$) (H is maximized at t to the value $-\lambda_0(t)$)
$\dot{\lambda} = -H_x$	differential equation satisfied by adjoint variables
$\dot{\lambda}_0 = -H_t$	differential equation satisfied by λ_0
σ, τ	directions such that $(x(T), T) + (\sigma, \tau)\epsilon$ is within $o(\epsilon)$ of a point in $\mathcal{S}$ for all sufficiently small $\epsilon > 0$ (transversality conditions are $(\lambda + K_x)^\top \sigma + (\lambda_0 + K_t)\tau = 0$)
$\Lambda(u)$	infinitesimal generator for a continuous time Markov process
$\lambda_{jk}(u)$	transition rate from state j to k in discrete state space Markov process
B	a constant such that $B > \sum_{k \neq j} \lambda_{jk}(u) > 0$, for all u and j
$p_{jk}(u)$	uniformized transition probabilities, i.e., $p_{jk} = \lambda_{jk}/(B + \alpha)$, $k \neq j$.
$B(t)$	Brownian motion
$g(x, u)$	coefficient for noise in stochastic differential equation, i.e., $\delta x = a(x, u)\delta t + g(x, u)\delta B$
N	$N = g^2$

Other notation as used in lectures 1–11, e.g., F, s, etc.