Part II. LQG Models

Lectures 7–11 have covered the LQG model and have thereby introduced various concepts, such as regulation, controllability, stabilizability, imperfect state observation, state estimation and certainty equivalence control. The ideas have been developed in discrete time, but some analogous results have been stated for continuous time also and have been used in examples. A good book for this part of the course is Whittle, *Optimization over Time, volume I*, Chapters 5, 17 and 18. The notation in lectures is fully consistent with this book.

As a result of studying this material you should be able to

- explain the meaning of the key terms used in lectures and listed overleaf.
- make use of the standard notation summarised overleaf.
- recall illustrative examples of various problem types (e.g., additive white noise, broom balancing, pendulum, satellite, dam, etc.)
- know the following results and conditions under which they hold:
  - Theorem 7.2 derivation of the LQ regulation optimal control and Riccati equation for cases with and without process noise; the fact that this can be extended to models with known disturbances and to tracking.
  - Section 7.5 idea of a linear model as linearisation about an equilibrium point.
  - Theorems 8.3, 8.4 necessary and sufficient conditions for controllability.
  - Sections 9.2, 9.3 necessary and sufficient conditions for stabilizability.
  - Lemma 9.1, Theorem 9.2 existence of the infinite horizon limit (but not the proof of 9.2).
  - Theorems 10.2, 10.3 necessary and sufficient conditions for observability.
  - Theorem 11.2 Kalman filtering.
  - Theorem 11.3 certainty equivalence and the separation principle.
- solve problems based on the LQG model that are similar to those in lectures and on Examples Sheet 2, by use of the dynamic programming equation, the Riccati equation, conditions for controllability, stabilizability and observability, ideas of Kalman filtering and certainty equivalence control.
- construct proofs based upon the following ideas:
  - Induction, (e.g., the Riccati equation for various problems with and without noise, and proofs of the Kalman filter and certainty equivalence principle.)
  - Linear algebra, (e.g., controllability, stabilizability and observability.)

Key terms in Lectures 7–11

Notation in Lectures 7–11

x_t, u_t, y_t	state, control and observation variables
n, m, r	dimensions of x_t, u_t, y_t
$[A, B, C]$	coefficient matrices of x_t, u_t, y_t in LQG plant and observation equations, $x_t = Ax_{t-1} + Bu_{t-1} + \epsilon_t$, $y_t = Cx_{t-1} + \eta_t$
R, S, Q	matrices in LQG one-step cost, $c(x, u) = x^\top Rx + u^\top Sx + x^\top S^\top u + u^\top Qu$
Π_h	matrix specifying LQG terminal cost, $\mathbf{K}(x_h) = x_h^\top \Pi_h x_h$
Π_t	matrix in LQG solution, $F(x, t) = x^\top \Pi_t x$, and satisfying Riccati recurrence, $\Pi_t = R + A^\top \Pi_{t+1} A - (S^\top + A^\top \Pi_{t+1} B)(Q + B^\top \Pi_{t+1} B)^{-1}(S + B^\top \Pi_{t+1} A)$
f	operator for backward Riccati recurrence, $\Pi_t = f\Pi_{t+1}$
K_t	matrix coefficient for optimal feedback control, $u_t = K_t x_t$, $K_t = -(Q + B^\top \Pi_{t+1} B)^{-1}(S + B^\top \Pi_{t+1} A)$
Γ_t	gain matrix, $\Gamma_t = A + BK_t$, such that $x_{t+1} = \Gamma_t x_t$
α_t	deterministic disturbances to plant equation, $x_t = Ax_{t-1} + Bu_{t-1} + \alpha_t$
σ_t	coefficient of linear term in solution to LQ regulation with disturbances
$\bar{x}_t, \bar{u}_t$	reference values for LQ tracking problem, $c(x, u) = (x - \bar{x}_t)^\top R(x - \bar{x}_t) + \cdots$
M_r	matrix for r-controllability, $M_r = [B\ AB\ \cdots\ A^{r-1}B]$
N_r	matrix for r-observability, $N_r = [C^\top\ (CA)^\top\ \cdots\ (CA^{r-1})^\top]^\top$
ϵ_t, η_t	plant and observation noise in LQG model
N, L, M	covariance matrices of noise, $N = E\left[\epsilon_t \epsilon_t^\top\right]$, $L = E\left[\epsilon_t \eta_t^\top\right]$, $M = E\left[\eta_t \eta_t^\top\right]$
V_{xy}	covariance of variables x and y
Y_t, U_t	observation and control histories at time t, $Y_t = (y_1, \ldots, y_t)$, $U_t = (u_0, \ldots, u_t)$
W_t	history available when choosing u_t, $W_t = (Y_t, U_{t-1})$
$\hat{x}_t$	LLS estimate of x_t, also $E(x_t \mid W_t)$ under Gaussian assumptions
Δ_t	estimation error, $\Delta_t = \hat{x}_t - x_t$
V_t	variance of $\hat{x}_t$ in Kalman filter, $E\left[\Delta_t \Delta_t^\top\right]$, $V_t = N + AV_{t-1}A^\top - (L + AV_{t-1}C^\top)(M + CV_{t-1}C^\top)^{-1}(L^\top + CV_{t-1}A^\top)$
$\tilde{y}_t$	innovation process of observations, used in Kalman filter, $\tilde{y}_t = y_t - C\hat{x}_{t-1}$
H_t	innovation coefficient matrix in Kalman filter, $\hat{x}_t = A\hat{x}_{t-1} + Bu_{t-1} + H_t(y_t - C\hat{x}_{t-1})$ $H_t = (L + AV_{t-1}C^\top)(M + CV_{t-1}C^\top)^{-1}$
$+\cdots$	policy independent terms in the minimal cost function, $F(\hat{x}, t) = \hat{x}^\top \Pi_t \hat{x} + \cdots$

Other notation as used in lectures 1–6, e.g., F, h, s, etc.