Optimization and Control: Examples Sheet 3

Continuous-time Models

1. [lecture 12] Consider the continuous-time system with scalar state variable, plant equation $\dot{x} = u$ and cost function $Q\int_0^h u^2 dt + x(h)^2$. By writing the DP equation in infinitesimal form and taking the appropriate limit, show that the value function satisfies

$$0 = \frac{\partial F}{\partial t} + \inf_u \left[Qu^2 + \frac{\partial F}{\partial x} u \right], \quad s > 0.$$

Show that F and the optimal control with time s to go are

$$F = \frac{Qx^2}{Q+s}, \qquad u = -\frac{x}{Q+s}.$$

2. [lecture 12] Consider the 'inertial' system (generalising Example 1) in which the two components of the state vector obey $\dot{x}_1 = x_2$ and $\dot{x}_2 = u$ and the cost function is $Q\int_0^h u^2 dt + x_1(h)^2$. Show that the value function and optimal control are

$$F_s = \frac{Q}{Q+s^3/3}(x_1 + sx_2)^2, \qquad u = -\frac{1}{Q+s^3/3} s(x_1+sx_2).$$

Note that this is consistent with the discrete-time assertion of the example on Examples Sheet 2. [Hint: Guess a solution of the form $F_s(x) = (x_1+sx_2)^2 \pi_s$.]

3. [lecture 12] Miss Prout holds the entire remaining stock of Cambridge elderberry wine for the vintage year 1959. If she releases it at rate u (in continuous time) she realises a unit price $p(u)$. She holds an amount x at time 0 and wishes to release this in such a way as to maximize her total discounted return, $\int_0^\infty e^{-\alpha t} up(u)\,dt$. Consider the particular case $p(u) = u^{-\gamma}$, where the constant γ is positive and less than one. Show that the value function is proportional to a power of x and determine the optimal release rule in closed-loop form (i.e., as a function of the present stock level.)

[Hint: The answers are $F(x) = (\gamma/\alpha)^\gamma x^{1-\gamma}$, $u = \alpha x/\gamma$. However, you should try to derive these answers from the DP equation; not simply substitute them into the DP equation and check that they work.]

4. [lecture 12] Let the vector x denote the Cartesian co-ordinates of a particle moving in $\mathbb{R}^d$. When at position x the particle moves with speed $v(x)$ and in a direction that can be chosen. The equation of motion is thus $\dot{x} = v(x)u$, where u is a unit vector to be chosen afresh at each position x. Let $F(x)$ denote the minimal time taken for the particle to reach a set $\mathcal{D}$ from a point x outside it. Show that after minimizing over u the dynamic programming equation for F implies that $|\nabla F(x)| = v(x)^{-1}$; i.e.,

$$\sum_{j=1}^d \left(\frac{\partial F}{\partial x_j} \right)^2 = v(x)^{-2}.$$

This is the *eikonal equation* of geometric optics; a short-wavelength form of the wave equation. How is the optimal direction at a given point determined from F?

[Hint: Show that the DP equation is $\inf_{u:|u|=1}[1 + v(x)u^\top \nabla F] = 0$. Then use a Cauchy-Schwartz inequality to show that the infimum is achieved by $u = -\nabla F/|\nabla F|$.]

5. [lecture 13] Consider the optimal control problem:

$$\text{minimize} \quad \int_0^T \tfrac{1}{2} u(t)^2\,dt \quad \text{subject to} \quad \dot{x}_1 = -x_1 + x_2\,, \quad \dot{x}_2 = -2x_2 + u\,,$$

where u is unrestricted, $x_1(0)$ and $x_2(0)$ are known, T is given and $x_1(T)$ and $x_2(T)$ are to be made to vanish. Rewrite the problem in terms of new variables, $z_1 = (x_1+x_2)e^t$ and $z_2 = x_2 e^{2t}$ and then show that the optimal control takes the form $u = \lambda_1 e^t + \lambda_2 e^{2t}$, for some constants λ_1 and λ_2. Find equations for $x_1(0)$, $x_2(0)$ in terms of λ_1, λ_2, and T, which you could in principle solve for λ_1, λ_2 in terms of $x_1(0)$, $x_2(0)$ and T.

Compare a linear feedback controller of the form $u(t) = -k_1 x_1(t) - k_2 x_2(t)$, where k_1 and k_2 are constants. Show that with this controller x_1 and x_2 cannot be made to vanish in finite time. Discuss the choice of optimal control with a quadratic performance criterion as opposed to linear feedback control, indicating which is likely to be more appropriate in given circumstances.

6. [lecture 13] A princess is jogging with speed r in the counterclockwise direction around a circular running track of radius r, and so has a position whose horizontal and vertical components at time t are $(r\cos t, r\sin t)$, $t \geq 0$. A monster who is initially located at the centre of the circle can move with velocity u_1 in the horizontal direction and u_2 in the vertical direction, where both velocities have a maximum magnitude of 1. The monster wishes to catch the princess in minimal time.

Analyse the monster's problem using Pontryagin's maximum principle. By considering feasible values for the adjoint variables, show that whatever the value of r the monster should always set at least one of $|u_1|$ or $|u_2|$ equal to 1. Show that if $r = \pi/\sqrt{8}$ then the monster catches the princess in minimal time by adopting the uniquely optimal policy $u_1 = 1$, $u_2 = 1$. Is the optimal policy always unique?

[Hint: Let x_1 and x_2 be the differences in the horizontal and vertical directions between the positions of the monster and princess.]

7. [lecture 14] In the neoclassical economic growth model, x is the existing capital per worker and u is consumption of capital per worker. The plant equation is

$$\dot{x} = f(x) - \gamma x - u, \tag{4}$$

where $f(x)$ is the production per worker, and $-\gamma x$ represents depreciation of capital and change in the size of the workforce. We wish to choose u to maximize

$$\int_{t=0}^T e^{-\alpha t} g(u)\,dt,$$

where $g(u)$ measures utility, is strictly increasing and concave, and T is prescribed. It is convenient to take a Hamiltonian

$$H = e^{-\alpha t}\left[g(u) + \lambda(f(x) - \gamma x - u)\right],$$

thereby including a discount factor in the definition of λ and expressing F in terms of present value.

Show that the optimal control satisfies $g'(u) = \lambda$ (assuming the maximum is at a stationary point) and

$$\dot{\lambda} = (\alpha + \gamma - f')\lambda. \tag{5}$$

Hence show that the optimal consumption obeys

$$\dot{u} = \frac{1}{\sigma(u)}[f'(x) - \alpha - \gamma], \quad \text{where} \quad \sigma(u) = -\frac{g''(u)}{g'(u)} > 0. \tag{6}$$

(σ is called the 'elasticity of marginal utility.')

Characterise an equilibrium solution, i.e., an $x(0) = \bar{x}$ such that the optimal trajectory is $x(t) = \bar{x}$, $t \geq 0$, and show that this $\bar{x}$ is independent of g.

8. [lecture 14] An aircraft flies in straight and level flight at height h, so that lift L balances weight mg. The mass rate of fuel consumption is proportional to the drag, and may be taken as $q = av^2 + b(Lv)^{-2}$, where a and b are constants and v is the speed. Thus

$$\dot{m} = -q = -av^2 - \frac{b}{(mgv)^2}.$$

Find a rule for determining v in terms of m if (i) the distance flown is to be maximized, (ii) the time spent flying at height h (until fuel is exhausted) is to be maximized.

$$\left[\text{Hint: Answers are (i) } v = \left(\frac{3b}{a(mg)^2}\right)^{1/4}, \text{ and (ii) } v = \left(\frac{b}{a(mg)^2}\right)^{1/4}\right].$$

9. [lecture 14] In what is known as Zermelo's problem, a straight river has current $c(y)$, where y is the distance from the bank from which a boat is leaving. The boat then crosses the river at constant speed v relative to the water, so that its position (x, y) satisfies $\dot{x} = v\cos\theta + c(y)$, $\dot{y} = v\sin\theta$, where θ is the heading angle indicated in the diagram.

(i) Suppose $c(y) > v$ for all y and the boatman wishes to be carried downstream as little as possible in crossing. Show that he should follow the heading

$$\theta = \cos^{-1}(-v/c(y)).$$

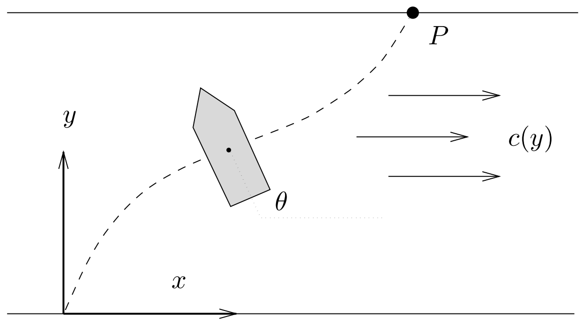

Figure 1: Zermelo's problem

(ii) Suppose the boatman wishes to reach a given point P on the opposite bank in minimal time. Show that he should follow the heading

$$\theta = \cos^{-1}\left(\frac{\lambda_1 v}{1 - \lambda_1 c(y)}\right),$$

where λ_1 is a parameter chosen to make his path pass through the target point.

10. [lecture 15] Customers arrive at a queue as a Poisson process of rate λ. They are served at rate $u = u(x)$, where x denotes the current size of the queue. Suppose that cost is incurred as rate $ax + bu$ where a, $b > 0$. The service rate u is the control variable. The dynamic programming equation in the infinite horizon limit is then

$$\gamma = \inf_u \left\{ax + bu(x) + \lambda[f(x+1) - f(x)] + u(x)1_{x>0}[f(x-1) - f(x)]\right\}$$

where γ denotes the average rate at which cost is incurred under the optimal policy and where $f(x)$ denotes the extra cost associated with starting from state x. (Here $1_{x>0} = 0$ if $x = 0$, and $1_{x>0} = 1$ if $x = 1, 2, 3, \ldots$.) Give a brief justification of this equation.

Show that under the constraint that u is a fixed constant, independent of x, and greater that λ then, for some C, there is a solution of the form

$$\gamma = \frac{a\lambda}{u - \lambda} + bu, \quad f(x) = C + \frac{ax(x+1)}{2(u - \lambda)}.$$

i.e., such that $f(x)$ does not grow exponentially in x (which is needed to ensure that $(1/t)Ef(x_t) \to 0$ as $t \to \infty$ and hence, similarly as in the proof for a discrete time model, that γ can be shown to be the time-average cost.) What is the optimal constant service rate?

Suppose now that we allow u to vary with x, subject to the constraint $m \leq u \leq M$, where $M > \lambda$. What is the policy which results if we carry out one stage of policy improvement to the optimal constant service policy?