Optimization and Control: Examples Sheet 2
LQG Models

1. [lecture 7] Consider a scalar deterministic linear system, $x_t = Ax_{t-1} + Bu_{t-1}$, with cost function $\sum_{t=0}^{h-1} Qu_t^2 + x_h^2$. Show from first principles (i.e., not simply by substituting values into the Riccati equation), that in terms of the time to go s, Π_s^{-1} obeys a linear recurrence and that

$$\Pi_s = \left[\frac{B^2}{Q(A^2-1)} + \left(1 - \frac{B^2}{Q(A^2-1)}\right) A^{-2s}\right]^{-1}.$$

Under what conditions does Π_s tend to a limit as $s \to \infty$? Investigate the limiting forms of Π_s and Γ_s.

2. [lecture 7] (88117) Successive attempts are made to regulate the speed of a clock, but each deliberate change in setting introduces also a random change whose size tends to increase with the size of the intended change. Explicitly, let x_n be the error in the speed of the clock after n corrections. On the basis of the observed value of x_n one attempts to correct the speed by an amount u_n. The actual error in speed then becomes

$$x_{n+1} = x_n - u_n + \epsilon_{n+1}$$

where, conditional on events up to the choice of u_n, the variable ϵ_{n+1} is normally distributed with zero mean and variance αu_n^2. If, after all attempts at regulation, one leaves the clock with an error x, then there is a cost x^2.

Suppose exactly s attempts are to be made to regulate the clock with initial error x. Determine the optimal policy and the minimal expected cost.

3. [lecture 7] Consider the scalar-state control problem with plant equation $x_{t+1} = x_t + u_t + \epsilon_t$ and cost function $\sum_{t=0}^{h-1} u_t^2 + Dx_h^2$. Here current state is observable, the horizon point h is prescribed, and the disturbances ϵ_t are i.i.d. with zero mean and variance v. Show that the open-loop form of the optimal control in the deterministic case $v = 0$ is $u_t = -Dx_0/(1 + hD)$ and that the closed-loop form of the optimal control is $u_t = -Dx_t/[1 + (h - t)D]$, whatever v.

Show that if the open-loop control is used in the stochastic case then a total expected cost $Dx_0^2/(1 + hD) + hDv$ is incurred, while use of the closed-loop control leads to a smaller expected cost of

$$F(x_0, 0) = \frac{Dx_0^2}{1 + hD} + Dv \sum_{s=0}^{h-1} \frac{1}{1 + sD}.$$

4. [lecture 7,11] (77314) Consider the real-valued system defined by

$$x_{n+1} = ax_n + \xi_n u_n \quad (n = 0, 1, \dots),$$

where u_t is the control at time n and $\{\xi_n; n = 0, 1, \dots\}$ is a sequence of independent random variables with mean b and variance σ^2. Suppose that the cost incurred at time n is $x_n^2 + u_n^2$, and that there are no terminal costs. Find the recursions satisfied by the finite-horizon optimal cost function. Is the optimal control certainty-equivalent control?

$$\left[\text{Hint: The answer is } F_s(x) = \Pi_s x^2, \text{ where } \Pi_s = 1 + \frac{a^2 \Pi_{s-1}(1 + \sigma^2 \Pi_{s-1})}{1 + (b^2 + \sigma^2)\Pi_{s-1}}.\right]$$

5. [lecture 8] Suppose that a discrete-time system with n-dimensional state variable x has a plant equation which is linear in state, $x_{t+1} = A_t x_t + b(u_t, t)$, an instantaneous cost $c(u_t, t)$ which is independent of state, and a terminal cost at time h that is a function of $d^\top x_h$, for a given vector d. Show that the value function takes the form $F(x, t) = \phi(\xi_t, t)$, where $\xi_t = d^\top z_t$ is the 'predicted miss distance' and $z_t = A_{h-1} \cdots A_t x_t$ is the the value that x_h would take if the system were uncontrolled from time t. Show that the optimal control at time t is also a function of ξ_t and t alone.

6. [lecture 8] (83117) Consider the linear system

$$\begin{aligned} x_{t+1} &= x_t + v_t \\ v_{t+1} &= v_t + u_t + \epsilon_t, \end{aligned}$$

where the state is the pair of scalars (x_t, v_t), representing the position and velocity of a body, $\{u_t\}$ is a sequence of control variables and $\{\epsilon_t\}$ is a sequence of independent zero-mean disturbances, with variance N. The objective is to minimize the expected value of $\sum_{t=0}^{T-1} u_t^2 + P_0 x_T^2$. Show that the optimal choice of u_t from state (x_t, v_t) is

$$u_t = -(s-1)P_s(x_t + sv_t),$$

where $s = T - t$ and

$$P_s^{-1} = P_0^{-1} + \tfrac{1}{6}s(s-1)(2s-1).$$

[Hint: use what you learned from Example 5 to reduce this problem to LQ regulation of a scalar quantity. Re-write the plant equation and cost in terms of this quantity and in terms of time to go.]

7. [lecture 8] (77129) A simple model of the rolling motion of a ship represents it as a damped simple pendulum driven by wave motion. For small roll angles the equation is

$$\ddot{\theta} + 2\gamma\omega\dot{\theta} + \omega^2\theta = \omega^2 u,$$

where $\theta(t)$ is the roll angle and $u(t)$ is the effective rolling torque from wave motion; ω and γ are constants.

Show that θ and $\dot{\theta}$ can in principle be moved from any values to any other values in an arbitrary short time by an appropriate u. State any theorems that you use.

8. [lecture8] (81117) A one-dimensional model of the problem faced by a juggler trying to balance a light stick with a weight on top is given by the equation

$$\ddot{x}_1 = \alpha(x_1 - u)$$

where x_1 is the horizontal displacement of the top of the stick from some fixed point and u is the horizontal displacement of the bottom. (The stick is assumed to be nearly upright and stationary and $\alpha > 0$ is inversely proportional to the length.) Show that the juggler can control x_1 by manipulating u.

If he tries to balance n such weighted sticks on top of one another, the equations governing stick k ($k = 2, \dots, n$) are (provided the weights on the sticks get smaller fast enough as n increases)

$$\ddot{x}_k = \alpha(x_k - x_{k-1})$$

Show that the n-stick system is controllable. [You may find it helpful to take the state vector as $(\dot{x}_1, x_1, \dot{x}_2, x_2, \dots, \dot{x}_n, x_n)^\top$. Example F.]

9. [lecture 11] (87117) Consider the controlled system $x_{t+1} = x_t + u_t + 3\epsilon_{t+1}$, where the ϵ_t are independent $N(0,1)$ variables. The instantaneous cost at time t is $x_t^2 + 2u_t^2$. Assuming that x_t is observable at time t, calculate the optimal control under steady-state (stationary) conditions and find the expected cost per unit time incurred when this control is used.

Suppose now that at time t one observes, not x_t, but $y_t = x_{t-1} + 2\eta_t$, where the η_t are again independent $N(0,1)$ variables independent of the ϵ_t. Show that the law of $\hat{x}_t$ conditional on $(y_1, \dots, y_t)$ has steady-state variance 12.

Determine the optimal control and a recursion for the optimal estimate of state under stationary conditions.

10. [lectures 9–11] (00414) Consider the system $x_{t+1} = Ax_t + Bu_t$, $x_t \in \mathbb{R}^n$, $u_t \in \mathbb{R}^m$, and let

$$F_t(x_0) = \min_{u_0,\dots,u_{t-1}} \sum_{s=0}^{t-1} x_s^\top R x_s \;\; + x_t^\top \Pi_0 x_t, ,$$

where R is positive definite. Assuming that the optimal control is of the form $u_s = K_s x_s$, and $F_t(x) = x^\top \Pi_t x$, show that

$$\Pi_t = f(R, A, B, \Pi_{t-1}) \equiv \min_K \left\{ R + (A+BK)^\top \Pi_{t-1}(A+BK) \right\} .$$

Explain what is meant by saying the system is controllable.

State necessary and sufficient condition for controllability in terms of A and B.

Show that if the system is controllable and $\Pi_0 = 0$, then $F_t(x)$ is monotone increasing in t and tends to the finite limit $x^\top \Pi x$, where $\Pi = f(R, A, B, \Pi)$.

Suppose now that $x_{t+1} = Ax_t + Bu_t + \epsilon_t$, where $\{\epsilon_t\}$ is noise, $E\epsilon_t = 0$, $E\epsilon_t\epsilon_t^\top = N$, and ϵ_s and ϵ_t are independent for $s \neq t$. Moreover, x_0 is known, but $x_1, x_2, \dots$ cannot be observed. Instead, we observe $y_1, y_2, \dots \in \mathbb{R}^r$, where $y_t = Cx_{t-1}$. Consider the estimate of x_t given by

$$\hat{x}_t = A\hat{x}_{t-1} + Bu_{t-1} - H_t(y_t - C\hat{x}_{t-1})$$

where $\hat{x}_0 = x_0$ and H_t is chosen to minimize, V_t, the covariance matrix of $\hat{x}_t$. Show that $\hat{x}_t$ is unbiased and that, with $V_0 = 0$,

$$V_t = f(N, A^\top, C^\top, V_{t-1}) = \min_H \left\{ N + (A+HC)V_{t-1}(A+HC)^\top \right\} .$$

Hence, quoting a condition in terms of A and C for the noiseless system to be observable, show that observability is a sufficient condition for V_t to tend to a finite limit as $t \to \infty$.