Connection between average-cost and discounted-cost with discount factor β near 1

Let $\{c_0, c_1, \dots\}$ be a sequence of costs. Assuming the limit exists, we can write the average value of this sequence as

$$\lim_{t\to\infty} \frac{1}{t}\sum_{\tau=0}^{t-1} c_\tau = \lim_{t\to\infty}\lim_{\beta\to 1} \frac{\sum_{\tau=0}^{t-1}\beta^\tau c_\tau}{\sum_{\tau=0}^{t-1}\beta^\tau}.$$

For example, if the sequence is $0,1,0,1,0,1,0,\dots$, the limit would be $1/2$. Suppose we can reverse the order of limits. This has the evaluation

$$\lim_{\beta\to 1}\lim_{t\to\infty} \frac{\sum_{\tau=0}^{t-1}\beta^\tau c_\tau}{\sum_{\tau=0}^{t-1}\beta^\tau} = \lim_{\beta\to 1}(1-\beta)\sum_{\tau=0}^{\infty}\beta^\tau c_\tau.$$

This show that if $F(x)$ is the infinite-horizon discounted sum of a sequence of costs $\{c_0, c_1, \dots\}$, obtained from starting in state x, with discount factor β, then $(1-\beta)F(x)$ tends to the average-cost as $\beta \to 1$.

If the state space and action spaces are finite, so that there are only a finite number of deterministic stationary Markov policies, then one such policy will be optimal for all β sufficiently close to 1, and that this policy also minimizes the average-cost.