Dynamic Programming and Optimal Control **Autumn 2014**

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Problems Sheet 4

1. Consider a scalar deterministic linear system, $x_t = Ax_{t-1}+Bu_{t-1}$, with cost function $\sum_{t=0}^{h-1} Qu_t^2+x_h^2$. Show from first principles (i.e., not simply by substituting values into the Riccati equation), that in terms of the time to go s, Π_s^{-1} obeys a linear recurrence and that

$$\Pi_s = \left[\frac{B^2}{Q(A^2-1)} + \left(1 - \frac{B^2}{Q(A^2-1)}\right) A^{-2s}\right]^{-1}.$$

Under what conditions does Π_s tend to a limit as $s \to \infty$? What are the limiting forms of Π_s and Γ_s?

2. Consider the controlled system $x_{t+1} = x_t+u_t+3\epsilon_{t+1}$, where the ϵ_t are independent $N(0,1)$ variables. The instantaneous cost at time t is $x_t^2 + 2u_t^2$. Assuming that x_t is observable at time t, show that the optimal control under steady-state (stationary) conditions is $u_t = -(1/2)x_t$, and that when this control is used the average-cost incurred per unit time is 18.

Suppose now that at time t one observes, not x_t, but $y_t = x_{t-1} + 2\eta_t$, where the η_t are again independent $N(0,1)$ variables independent of the ϵ_t. The law of $\hat{x}_t$ conditional on $(y_1,\ldots,y_t)$ is Gaussian, with a mean that is a the linear function of $\hat{x}_t$, u_t and y_{t+1} having minimum variance. Find under steady-state conditions this linear function, and show that $\hat{x}_t$ has steady-state variance 12.

Assuming stationary conditions, express the optimal control, u_t, as a function of $\hat{x}_t$.

3. Miss Prout holds the entire remaining stock of Cambridge elderberry wine for the vintage year 1959. If she releases it at rate u (in continuous time) she realises a unit price $p(u)$. She holds an amount x at time 0 and wishes to release this in such a way as to maximize her total discounted return, $\int_0^\infty e^{-\alpha t} up(u)\,dt$. Consider the particular case $p(u) = u^{-\gamma}$, where the constant γ is positive and less than one. Show that the value function is proportional to a power of x and determine the optimal release rule in closed-loop form (i.e., as a function of the present stock level.)

[Hint: The answers are $F(x) = (\gamma/\alpha)^\gamma x^{1-\gamma}$, $u = \alpha x/\gamma$. However, you should try to derive these answers from the DP equation; not simply substitute them into the DP equation and check that they work.]

4. Consider the optimal control problem:

$$\text{minimize} \quad \int_0^T \tfrac{1}{2}u(t)^2\,dt \quad \text{subject to} \quad \dot{x}_1 = -x_1 + x_2\,, \quad \dot{x}_2 = -2x_2 + u\,,$$

where u is unrestricted, $x_1(0)$ and $x_2(0)$ are known, T is given and $x_1(T)$ and $x_2(T)$ are to be made to vanish. Rewrite the problem in terms of new variables, $z_1 = (x_1+x_2)e^t$ and $z_2 = x_2e^{2t}$ and then show that the optimal control takes the form $u = \lambda_1 e^t + \lambda_2 e^{2t}$, for some constants λ_1 and λ_2. Find equations for $x_1(0)$, $x_2(0)$ in terms of λ_1, λ_2, and T, which you could in principle solve for λ_1, λ_2 in terms of $x_1(0)$, $x_2(0)$ and T.

Compare a linear feedback controller of the form $u(t) = -k_1x_1(t) - k_2x_2(t)$, where k_1 and k_2 are constants. Show that with this controller x_1 and x_2 cannot be made to vanish in finite time. Discuss the choice of optimal control with a quadratic performance criterion as opposed to linear feedback control, indicating which is likely to be more appropriate in given circumstances.

5. A princess is jogging with speed r in the counterclockwise direction around a circular running track of radius r, and so has a position whose horizontal and vertical components at time t are $(r\cos t, r\sin t)$, $t \geq 0$. A monster who is initially located at the centre of the circle can move with velocity u_1 in the horizontal direction and u_2 in the vertical direction, where both velocities have a maximum magnitude of 1. The monster wishes to catch the princess in minimal time.

Analyse the monster's problem using Pontryagin's maximum principle. By considering feasible values for the adjoint variables, show that whatever the value of r the monster should always set at least one of $|u_1|$ or $|u_2|$ equal to 1. Show that if $r = \pi/\sqrt{8}$ then the monster catches the princess in minimal time by adopting the uniquely optimal policy $u_1 = 1$, $u_2 = 1$. Is the optimal policy always unique?

[Hint: Let x_1 and x_2 be the differences in the horizontal and vertical directions between the positions of the monster and princess.]

6. In the neoclassical economic growth model, x is the existing capital per worker and u is consumption of capital per worker. The plant equation is

$$\dot{x} = f(x) - \gamma x - u, \tag{3}$$

where $f(x)$ is the production per worker, and $-\gamma x$ represents depreciation of capital and change in the size of the workforce. We wish to choose u to maximize

$$\int_{t=0}^{T} e^{-\alpha t} g(u)\, dt,$$

where $g(u)$ measures utility, is strictly increasing and concave, and T is prescribed. It is convenient to take a Hamiltonian

$$H = e^{-\alpha t}\left[g(u) + \lambda(f(x) - \gamma x - u)\right],$$

thereby including a discount factor in the definition of λ and expressing F in terms of present value.

Show that the optimal control satisfies $g'(u) = \lambda$ (assuming the maximum is at a stationary point) and

$$\dot{\lambda} = (\alpha + \gamma - f')\lambda. \tag{4}$$

Hence show that the optimal consumption obeys

$$\dot{u} = \frac{1}{\sigma(u)}[f'(x) - \alpha - \gamma], \qquad \text{where} \qquad \sigma(u) = -\frac{g''(u)}{g'(u)} > 0. \tag{5}$$

(σ is called the 'elasticity of marginal utility.')

Characterise an equilibrium solution, i.e., an $x(0) = \bar{x}$ such that the optimal trajectory is $x(t) = \bar{x}$, $t \geq 0$, and show that this $\bar{x}$ is independent of g.

7. An aircraft flies in straight and level flight at height h, so that lift L balances weight mg. The mass rate of fuel consumption is proportional to the drag, and may be taken as $q = av^2 + b(Lv)^{-2}$, where a and b are constants and v is the speed. Thus

$$\dot{m} = -q = -av^2 - \frac{b}{(mgv)^2}.$$

Find a rule for determining v in terms of m if (i) the distance flown is to be maximized, (ii) the time spent flying at height h (until fuel is exhausted) is to be maximized.

$$\left[\text{Hint: Answers are (i) } v = \left(\frac{3b}{a(mg)^2}\right)^{1/4}, \text{ and (ii) } v = \left(\frac{b}{a(mg)^2}\right)^{1/4}\right].$$

8. In Zermelo's navigation problem (proposed in 1931) a straight river has current $c(y)$, where y is the distance from the bank from which a boat is leaving. A boat is to cross the river at constant speed v relative to the water, so that its position (x, y) satisfies $\dot{x} = v\cos\theta + c(y)$, $\dot{y} = v\sin\theta$, where θ is the heading angle indicated in the diagram.

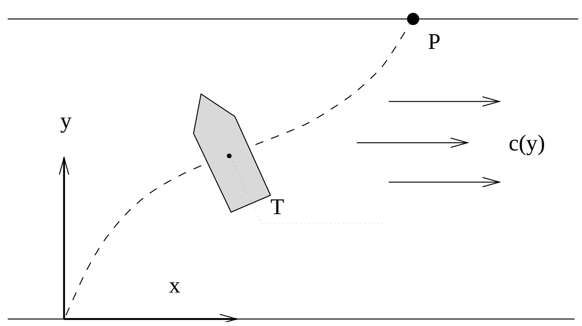

Figure 1: Zermelo's navigation problem

(i) Suppose $c(y) > v$ for all y and the boatman wishes to be carried downstream as little as possible in crossing. Show that he should follow the heading

$$\theta = \cos^{-1}(-v/c(y)).$$

(ii) Suppose the boatman wishes to reach a given point P on the opposite bank in minimal time. Show that he should follow the heading

$$\theta = \cos^{-1}\left(\frac{\lambda_1 v}{1 - \lambda_1 c(y)}\right),$$

where λ_1 is a parameter chosen to make his path pass through the target point.