16 Controlled Diffusion Processes

We give a brief introduction to controlled continuous-time stochastic models with a continuous state space, i.e., controlled diffusion processes.

16.1 Diffusion processes and controlled diffusion processes

The **Wiener process** $\{B(t)\}$, is a scalar process for which $B(0) = 0$, the increments in B over disjoint time intervals are statistically independent and $B(t)$ is normally distributed with zero mean and variance t. ('B' stands for **Brownian motion**.) This specification is internally consistent because, for example,

$$B(t) = B(t_1) + [B(t) - B(t_1)]$$

and for $0 \le t_1 \le t$ the two terms on the right-hand side are independent normal variables of zero mean and with variance t_1 and $t - t_1$ respectively.

If δB is the increment of B in a time interval of length δt then

$$E(\delta B) = 0\,, \quad E[(\delta B)^2] = \delta t\,, \quad E[(\delta B)^j] = o(\delta t)\,, \quad \text{for } j > 2\,,$$

where the expectation is one conditional on the past of the process. Note that since

$$E[(\delta B/\delta t)^2] = O\left[(\delta t)^{-1}\right] \to \infty\,,$$

the formal derivative $\epsilon = dB/dt$ (continuous-time 'white noise') does not exist in a mean-square sense, but expectations such as

$$E\left[\left\{\int \alpha(t)\epsilon(t)dt\right\}^2\right] = E\left[\left\{\int \alpha(t)dB(t)\right\}^2\right] = \int \alpha(t)^2 dt$$

make sense if the integral is convergent.

Now consider a **stochastic differential equation**

$$\delta x = a(x,u)\delta t + g(x,u)\delta B\,,$$

which we shall write formally as

$$\dot{x} = a(x,u) + g(x,u)\epsilon\,.$$

This, as a Markov process, has an infinitesimal generator with action

$$\begin{aligned}\Lambda(u)\phi(x) &= \lim_{\delta t \downarrow 0} E\left[\frac{\phi(x(t+\delta t)) - \phi(x)}{\delta t}\,\middle|\, x(t) = x, u(t) = u\right]\\ &= \phi_x a + \tfrac{1}{2}\phi_{xx} g^2\\ &= \phi_x a + \tfrac{1}{2}N\phi_{xx}\,,\end{aligned}$$

where $N(x,u) = g(x,u)^2$. So this **controlled diffusion process** has DP equation

$$\inf_u \left[c + F_t + F_x a + \tfrac{1}{2}N F_{xx}\right] = 0\,, \tag{16.1}$$

and in the vector case

$$\inf_u \left[c + F_t + F_x a + \tfrac{1}{2}\mathrm{tr}(N F_{xx})\right] = 0\,.$$

16.2 Example: noisy LQ regulation in continuous time

The dynamic programming equation is

$$\inf_u \left[x^\top R x + u^\top Q u + F_t + F_x^\top (Ax + Bu) + \tfrac{1}{2}\mathrm{tr}(N F_{xx})\right] = 0\,.$$

In analogy with the discrete and deterministic continuous cases that we have considered previously, we try a solution of the form,

$$F(x,t) = x^\top \Pi(t) x + \gamma(t)\,.$$

This leads to the same Riccati equation as in Section 12.2,

$$0 = x^\top \left[R + \Pi A + A^\top \Pi - \Pi B Q^{-1} B^\top \Pi + \frac{d\Pi}{dt}\right] x\,,$$

and also, as in Section 7.3,

$$\frac{d\gamma}{dt} + \mathrm{tr}(N\Pi(t)) = 0, \quad \text{giving} \quad \gamma(t) = \int_t^T \mathrm{tr}(N\Pi(\tau))\,d\tau\,.$$

16.3 Example: a noisy second order system

Consider a special case of LQ regulation:

$$\operatorname*{minimize}_u E\left[x(T)^2 + \int_0^T u(t)^2\,dt\right]$$

where for $0 \le t \le T$,

$$\dot{x}(t) = y(t) \quad \text{and} \quad \dot{y}(t) = u(t) + \epsilon(t)\,,$$

$u(t)$ is the control variable, and $\epsilon(t)$ is **Gaussian white noise**,

Note that if we define $z(t) = x(t) + (T-t)y(t)$ then

$$\dot{z} = \dot{x} - y + (T-t)\dot{y} = (T-t)u + (T-t)\epsilon(t)$$

where $z(T) = x(T)$. Hence the problem can be posed only in terms of scalars u and z.

Recalling what we know about LQ models, let us conjecture that the optimality equation is of a form

$$V(z,t) = z^2 P_t + \gamma_t \,. \tag{16.2}$$

We could use (16.1). But let us argue from scratch. For (16.2) to work we will need

$$\begin{aligned} z^2 P_t + \gamma_t &= \min_u \left\{ u^2\delta + E\left[(z+\dot z\delta)^2 P_{t+\delta} + \gamma_{t+\delta}\right]\right\} \\ &= \min_u \left\{ u^2\delta + \left[z^2 + 2(T-t)zu\delta + (T-t)^2\delta\right] P_{t+\delta} + \gamma_{t+\delta}\right\} + o(\delta) \end{aligned}$$

The optimizing u is

$$u = -(T-t)P_{t+\delta} z \,.$$

Substituting this and letting $\delta \to 0$ we have

$$-z^2 \dot P_t - \dot\gamma_t = -z^2 (T-t)^2 P_t^2 + (T-t)^2 P_t \,.$$

Thus

$$-\dot\gamma_t = (T-t)^2 P_t$$

and

$$\dot P_t = (T-t)^2 P_t^2 \,.$$

Using the boundary condition $P_T = 1$, we find that the solution to the above differential equation is

$$P_t = \left(1 + \tfrac{1}{3}(T-t)^3\right)^{-1} ,$$

and the optimal control is

$$u(t) = -(T-t)\left(1 + \tfrac{1}{3}(T-t)^3\right)^{-1} z(t) \,.$$

16.4 Example: passage to a stopping set

Consider a problem of movement on the unit interval $0 \le x \le 1$ in continuous time, $\dot x = u + \epsilon$, where ϵ is white noise of **power** v. The process terminates at time T when x reaches one end or the other of the the interval. The cost is made up of an integral term $\frac{1}{2}\int_0^T (L + Qu^2)dt$, penalising both control and time spent, and a terminal cost which takes the value C_0 or C_1 according as termination takes place at 0 or 1.

Show that in the deterministic case $v = 0$ one should head straight for one of the termination points at a constant rate and that the value function $F(x)$ has a piecewise linear form, with possibly a discontinuity at one of the boundary points if that boundary point is the optimal target from no interior point of the interval.

Show, in the stochastic case, that the dynamic programming equation with the control value optimized out can be linearised by a transformation $F(x) = \alpha \log \phi(x)$ for a suitable constant α, and hence solve the problem.

Solution. In the deterministic case the optimality equation is

$$\inf_u \left[\frac{L+Qu^2}{2} + u\frac{\partial F}{\partial x}\right] = 0, \quad 0 < x < 1, \tag{16.3}$$

with boundary conditions $F(0) = C_0$, $F(1) = C_1$. If one goes (from x) for $x = 0$ at speed w one incurs a cost of $C_0 + (x/2w)(L + Qw^2)$ with a minimum over w value of $C_0 + x\sqrt{LQ}$. Indeed (16.3) is solved by

$$F(x) = \min\left[C_0 + x\sqrt{LQ}, C_1 + (1-x)\sqrt{LQ}\right].$$

The minimizing option determines the target and the optimal w is $\sqrt{L/Q}$.

In the stochastic case

$$\inf_u \left[\frac{L+Qu^2}{2} + u\frac{\partial F}{\partial x} + \frac{v}{2}\frac{\partial^2 F}{\partial x^2}\right] = 0.$$

So $u = -Q^{-1}F_x$ and

$$L - Q^{-1}\left(\frac{\partial F}{\partial x}\right)^2 + v\frac{\partial^2 F}{\partial x^2} = 0 \,.$$

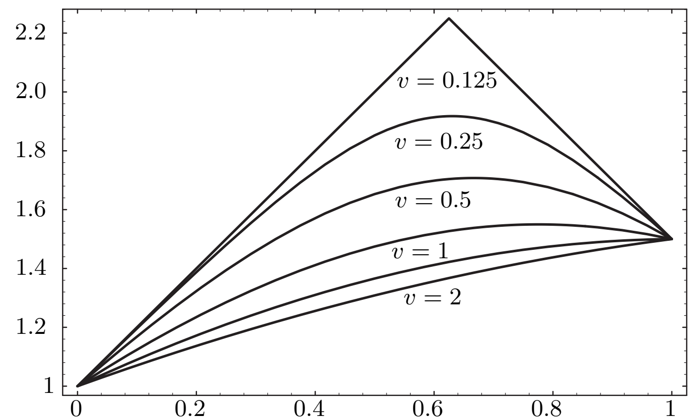
Make the transform $F(x) = -Qv \log\phi(x)$ so $\phi(x) = e^{-F(x)/Qv}$. Then

$$Qv^2 \frac{\partial^2 \phi}{\partial x^2} - L\phi = 0 \,,$$

with solution

$$\phi(x) = k_1 \exp\left(\frac{x}{v}\sqrt{L/Q}\right) + k_2 \exp\left(-\frac{x}{v}\sqrt{L/Q}\right) .$$

We choose the constants k_1, k_2 to meet the two boundary conditions on F.



$F(x)$ against x for the passage to a stopping set

The figure shows the solution for $L = 1$, $Q = 4$, $C_0 = 1$, $C_1 = 1.5$ and $v = 0.125$, 0.25, 0.5, 1, 2 and the deterministic solution. Notice that noise actually reduces cost by lessening the time until absorption at one or the other of the endpoints. ■