14 Applications of the Maximum Principle

We discuss the terminal conditions of the maximum principle and further examples of its use. The arguments are typical of those used to synthesise a solution to an optimal control problem by use of the maximum principle.

14.1 Problems with terminal conditions

Suppose a, c, $\mathcal{S}$ and $\mathbf{K}$ are all t-dependent. The DP equation for $F(x,t)$ is now be

$$\inf_u [c + F_t + F_x a] = F_t - \sup_u [\lambda^\top a - c] = 0\,, \tag{14.1}$$

outside a stopping set $\mathcal{S}$, with $F(x,t) = \mathbf{K}(x,t)$ for (x,t) in $\mathcal{S}$. However, we can reduce this to a formally time-invariant case by augmenting the state variable x by the variable t. We then have the augmented variables

$$x \to \begin{bmatrix} x \\ t \end{bmatrix} \quad a \to \begin{bmatrix} a \\ 1 \end{bmatrix} \quad \lambda \to \begin{bmatrix} \lambda \\ \lambda_0 \end{bmatrix}.$$

We keep the same definition (13.4) as before, that $H = \lambda^\top a - c$, and take $\lambda_0 = -F_t$. It now follows from (14.1) that on the optimal trajectory

$$H(x,u,\lambda) \text{ is maximized to } -\lambda_0\,.$$

Theorem 13.1 still holds, as can be verified. However, to (13.6) we can now add

$$\dot\lambda_0 = -H_t = c_t - \lambda a_t\,. \tag{14.2}$$

and transversality condition

$$(\lambda + \mathbf{K}_x)^\top \sigma + (\lambda_0 + \mathbf{K}_t)\tau = 0\,, \tag{14.3}$$

which must hold at the termination point (x,t) if $(x + \epsilon\sigma, t + \epsilon\tau)$ is within $o(\epsilon)$ of the termination point of an optimal trajectory for all small enough positive ϵ. We can now understand what to do with various types of terminal condition.

If the stopping rule specifies only a **fixed terminal time** T then τ must be zero and σ is unconstrained, so that (14.3) becomes $\lambda(T) = -\mathbf{K}_x$. The problem in Section 13.4 is like this.

If there is a **free terminal time** then τ is unconstrained and so (14.3) gives $-\lambda_0(T) = \mathbf{K}_T$. An example of this case appears in Section 14.2 below.

If the system is time-homogeneous, in that a and c are independent of t, but the terminal cost $\mathbf{K}(x,T)$ depends on T, then (14.2) implies that λ_0 is constant and so the maximized value of H is constant on the optimal orbit. The problem in Section 13.2 could have been solved this way by replacing $C = \int_0^T 1\,dt$ by $C = \mathbf{K}(x,T) = T$. We would deduce from the transversality condition that since τ is unconstrained, $\lambda_0 = -\mathbf{K}_T = -1$. Thus $H = \lambda_1 x_2 + \lambda_2 u$ is maximized to 1 at all points of the optimal trajectory.

14.2 Example: monopolist

Miss Prout holds the entire remaining stock of Cambridge elderberry wine for the vintage year 1959. If she releases it at rate u (in continuous time) she realises a unit price $p(u) = (1 - u/2)$, for $0 \le u \le 2$ and $p(u) = 0$ for $u \ge 2$. She holds an amount x at time 0 and wishes to release it in a way that maximizes her total discounted return, $\int_0^T e^{-\alpha t} u p(u)\,dt$, (where T is unconstrained.)

Solution. The plant equation is $\dot x = -u$ and the Hamiltonian is

$$H(x,u,\lambda) = e^{-\alpha t} u p(u) - \lambda u = e^{-\alpha t} u(1 - u/2) - \lambda u\,.$$

Note that $\mathbf{K} = 0$. Maximizing with respect to u and using $\dot\lambda = -H_x$ gives

$$u = 1 - \lambda e^{\alpha t}\,, \qquad \dot\lambda = 0\,, \qquad t \ge 0\,,$$

so λ is constant. The terminal time is unconstrained so the transversality condition gives $\lambda_0(T) = -\mathbf{K}_T = 0$. Therefore, since H is maximized to $-\lambda_0(T) = 0$ at T, we have $u(T) = 0$ and hence

$$\lambda = e^{-\alpha T}\,, \qquad u = 1 - e^{-\alpha(T-t)}\,, \quad t \le T\,,$$

where T is then the time at which all wine has been sold, and so

$$x = \int_0^T u\,dt = T - \left(1 - e^{-\alpha T}\right)/\alpha\,.$$

Thus u is implicitly a function of x, through T. The optimal value function is

$$F(x) = \int_0^T (u - u^2/2) e^{-\alpha t}\,dt = \frac{1}{2}\int_0^T \left(e^{-\alpha t} - e^{\alpha t - 2\alpha T}\right) dt = \frac{\left(1 - e^{-\alpha T}\right)^2}{2\alpha}\,.$$

■

14.3 Example: insects as optimizers

A colony of insects consists of workers and queens, of numbers $w(t)$ and $q(t)$ at time t. If a time-dependent proportion $u(t)$ of the colony's effort is put into producing workers, $(0 \le u(t) \le 1$, then w, q obey the equations

$$\dot w = auw - bw, \quad \dot q = c(1-u)w\,,$$

where a, b, c are constants, with $a > b$. The function u is to be chosen to maximize the number of queens at the end of the season. Show that the optimal policy is to produce only workers up to some moment, and produce only queens thereafter.

Solution. The Hamiltonian is

$$H = \lambda_1(auw - bw) + \lambda_2 c(1-u)w\,.$$

The adjoint equations and transversality conditions (with $\mathbf{K} = -q$) give

$$\begin{array}{rll} -\dot\lambda_0 & = H_t = & 0 \\ -\dot\lambda_1 & = H_w = & \lambda_1(au-b)+\lambda_2 c(1-u) \\ -\dot\lambda_2 & = H_q = & 0 \end{array}, \qquad \begin{array}{rll} \lambda_1(T) & = -K_w = & 0 \\ \lambda_2(T) & = -K_q = & 1 \end{array},$$

and hence $\lambda_0(t)$ is constant and $\lambda_2(t) = 1$ for all t. Therefore H is maximized by

$$u = \begin{matrix} 0 \\ 1 \end{matrix} \quad \text{as } \lambda_1 a - c \quad \begin{matrix} \le \\ \ge \end{matrix} \quad 0\,.$$

At T, this implies $u(T) = 0$. If t is a little less than T, λ_1 is small and $u = 0$ so the equation for λ_1 is

$$\dot\lambda_1 = \lambda_1 b - c. \tag{14.4}$$

As long as λ_1 is small, $\dot\lambda_1 < 0$. Therefore as the *remaining time* s increases, $\lambda_1(s)$ increases, until such point that $\lambda_1 a - c \ge 0$. The optimal control becomes $u = 1$ and then $\dot\lambda_1 = -\lambda_1(a-b) < 0$, which implies that $\lambda_1(s)$ continues to increase as s increases, right back to the start. So there is no further switch in u.

The point at which the single switch occurs is found by integrating (14.4) from t to T, to give $\lambda_1(t) = (c/b)(1 - e^{-(T-t)b})$ and so the switch occurs where $\lambda_1 a - c = 0$, i.e., $(a/b)(1 - e^{-(T-t)b}) = 1$, or

$$t_{\text{switch}} = T + (1/b)\log(1 - b/a).$$

Experimental evidence suggests that social insects do closely follow this policy and adopt a switch time that is nearly optimal for their natural environment. ■

14.4 Example: rocket thrust optimization

Regard a rocket as a point mass with position x, velocity v and mass m. Mass is changed only by expansion of matter in the jet. Suppose the jet has vector velocity k relative to the rocket and the rocket is subject to external force f. Then the condition of momentum conservation yields

$$(m - \delta m)(v + \delta v) + (v - k)\delta m - mv = f\delta t\,,$$

and this gives the so-called 'rocket equation',

$$m\dot v = k\dot m + f\,.$$

Suppose the jet speed $|k| = 1/b$ is fixed, but the direction and the rate of expulsion of mass can be varied. Then the control is the thrust vector $u = k\dot m$, subject to $|u| \le 1$, say. Find the control that maximizes the height that the rocket reaches.

Solution. The plant equation (in $\mathbb{R}^3$) is

$$\begin{aligned} \dot x &= v \\ m\dot v &= u + f \\ \dot m &= -b|u|\,. \end{aligned}$$

We take dual variables p, q, r corresponding to x, v, m. Then

$$H = p^\top v + \frac{q^\top(u+f)}{m} - rb|u| - c\,,$$

(where if the costs are purely terminal $c = 0$), and u must maximize

$$\frac{q^\top u}{m} - br|u|.$$

The optimal u is in the direction of q so $u = |u|q/|q|$ and $|u|$ maximizes

$$|u|\left(\frac{|q|}{m} - br\right).$$

Thus we have that the optimal thrust should be

$$\begin{matrix} \text{maximal} \\ \text{intermediate} \\ \text{null} \end{matrix} \quad \text{as} \quad \left(\frac{|q|}{m} - rb\right) \quad \begin{matrix} > \\ = \\ < \end{matrix} \quad 0\,.$$

The control is bang/bang and p, q, r are determined from the dual equations.

If the rocket is launched vertically then $f = -mg$ and the dual equations give $\dot p = 0$, $\dot q = -p$ and $\dot r = qu/m^2 > 0$. Suppose we want to maximize the height that the rocket attains. Let m_0 be the mass of the rocket structure, so that the maximum height has been reached if $m = m_0$ and $v \le 0$. Since $\mathbf{K} = -x$ at termination, the transversality conditions give $p(T) = 1$, $q(T) = 0$. Thus $p(s) = 1$, $q(s) = s$, and $|u|$ must maximize $|u|\,(s/m - br)$. One can check that $(d/ds)(s/m - rb) > 0$, and hence we should use full thrust from launch up to some time, and thereafter coast to maximum height on zero thrust. ■