8 Controllability

We define and give conditions for controllability in discrete and continuous time.

8.1 Controllability

Consider the $[A, B, \cdot\,]$ system with plant equation $x_{t+1} = Ax_t + u_t$. The **controllability** question is: can we bring x to an arbitrary prescribed value by some u-sequence?

Definition 8.1 *The system is* ***r-controllable*** *if one can bring it from an arbitrary prescribed x_0 to an arbitrary prescribed x_r by some u-sequence $u_0, u_1, \dots, u_{r-1}$. A system of dimension n is said to be* **controllable** *if it is r-controllable for some r*

Example. If B is square and non-singular then the system is 1-controllable, for

$$x_1 = Ax_0 + Bu_0 \quad \text{where} \quad u_0 = B^{-1}(x_1 - Ax_0)\,.$$

Example. Consider the case, $(n = 2,\ m = 1)$,

$$x_t = \begin{pmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{pmatrix} x_{t-1} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u_{t-1}.$$

This system is not 1-controllable. But

$$x_2 - A^2 x_0 = Bu_1 + ABu_0 = \begin{pmatrix} 1 & a_{11} \\ 0 & a_{21} \end{pmatrix} \begin{pmatrix} u_1 \\ u_0 \end{pmatrix}.$$

So it is 2-controllable if and only if $a_{21} \neq 0$.

More generally, by substituting the plant equation into itself, we see that we must find $u_0, u_1, \dots, u_{r-1}$ to satisfy

$$\Delta = x_r - A^r x_0 = Bu_{r-1} + ABu_{r-2} + \cdots + A^{r-1}Bu_0, \tag{8.1}$$

for arbitrary Δ. In providing conditions for controllability we shall need to make use of the following theorem.

Theorem 8.2 (The Cayley-Hamilton theorem) *Any $n \times n$ matrix A satisfies its own characteristic equation. So that if*

$$\det(\lambda I - A) = \sum_{j=0}^{n} a_j \lambda^{n-j}$$

then

$$\sum_{j=0}^{n} a_j A^{n-j} = 0. \tag{8.2}$$

The implication is that $I, A, A^2, \dots, A^{n-1}$ contains basis for A^r, $r = 0, 1, \dots$. Proof. (*starred*) Define

$$\Phi(z) = \sum_{j=0}^{\infty} (Az)^j = (I - Az)^{-1} = \frac{\operatorname{adj}(I - Az)}{\det(I - Az)}.$$

Then

$$\det(I - Az)\Phi(z) = \sum_{j=0}^{n} a_j z^j \Phi(z) = \operatorname{adj}(I - Az),$$

which implies (8.2) since the coefficient of z^n must be zero. ■

We are now in a position to characterise controllability.

Theorem 8.3 *(i) The system $[A, B, \cdot\,]$ is r-controllable if and only if the matrix*

$$M_r = \begin{bmatrix} B & AB & A^2B & \cdots & A^{r-1}B \end{bmatrix}$$

has rank n, or (ii) equivalently, if and only if the $n \times n$ matrix

$$M_r M_r^\top = \sum_{j=0}^{r-1} A^j (BB^\top)(A^\top)^j$$

is nonsingular (or, equivalently, positive definite.) (iii) If the system is r-controllable then it is s-controllable for $s \geq \min(n, r)$, and (iv) a control transferring x_0 to x_r with minimal cost $\sum_{t=0}^{r-1} u_t^\top u_t$ is

$$u_t = B^\top (A^\top)^{r-t-1} (M_r M_r^\top)^{-1} (x_r - A^r x_0), \quad t = 0, \dots, r-1.$$

Proof. (i) The system (8.1) has a solution for arbitrary Δ if and only if M_r has rank n. (ii) $M_r M_r^\top$ is singular if and only if there exists w such that $M_r M_r^\top w = 0$, and

$$M_r M_r^\top w = 0 \Longleftrightarrow w^\top M_r M_r^\top w = 0 \Longleftrightarrow M_r^\top w = 0.$$

(iii) The rank of M_r is non-decreasing in r, so if it is r-controllable, then it is s-controllable for $s \geq r$. But the rank is constant for $r \geq n$ by the Cayley-Hamilton theorem. (iv) Consider the Lagrangian

$$\sum_{t=0}^{r-1} u_t^\top u_t + \lambda^\top \Big(\Delta - \sum_{t=0}^{r-1} A^{r-t-1} B u_t\Big),$$

giving

$$u_t = \tfrac{1}{2} B^\top (A^\top)^{r-t-1} \lambda.$$

Now we can determine λ from (8.1). ■

8.2 Controllability in continuous-time

Theorem 8.4 *(i) The n dimensional system $[A, B, \ \cdot\ ]$ is controllable if and only if the matrix M_n has rank n, or (ii) equivalently, if and only if*

$$G(t) = \int_0^t e^{As} BB^\top e^{A^\top s}\, ds,$$

is positive definite for all $t > 0$. (iii) If the system is controllable then a control that achieves the transfer from $x(0)$ to $x(t)$ with minimal control cost $\int_0^t u_s^\top u_s ds$ is

$$u(s) = B^\top e^{A^\top (t-s)} G(t)^{-1}(x(t) - e^{At}x(0)).$$

Note that there is now no notion of r-controllability. However, $G(t) \downarrow 0$ as $t \downarrow 0$, so the transfer becomes more difficult and costly as $t \downarrow 0$.

8.3 Example: broom balancing

Consider the problem of balancing a broom in an upright position on your hand. By Newton's laws, the system obeys $m(\ddot{u}\cos\theta + L\ddot{\theta}) = mg\sin\theta$. For small θ we have $\cos\theta \sim 1$ and $\theta \sim \sin\theta = (x-u)/L$, so with $\alpha = g/L$ the plant equation is

$$\ddot{x} = \alpha(x - u),$$

equivalently,

$$\frac{d}{dt}\begin{pmatrix} x \\ \dot{x} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \alpha & 0 \end{pmatrix}\begin{pmatrix} x \\ \dot{x} \end{pmatrix} + \begin{pmatrix} 0 \\ -\alpha \end{pmatrix} u\,.$$

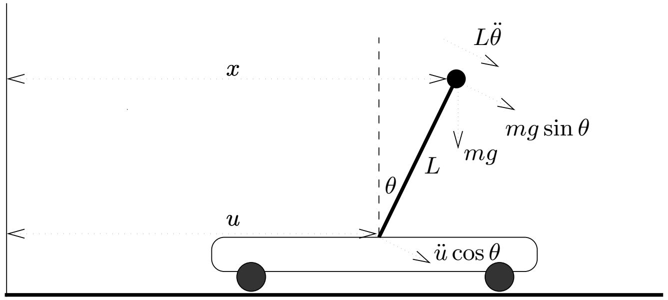

Figure 1: Force diagram for broom balancing

Since

$$\begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & -\alpha \\ -\alpha & 0 \end{bmatrix},$$

the system is controllable if θ is initially small.

8.4 Example: satellite in a plane orbit

Consider a satellite of unit mass in a planar orbit and take polar coordinates (r, θ).

$$\ddot{r} = r\dot{\theta}^2 - \frac{c}{r^2} + u_r, \qquad \ddot{\theta} = -\frac{2\dot{r}\dot{\theta}}{r} + \frac{1}{r}u_\theta\,,$$

where u_r and u_θ are the radial and tangential components of thrust. If $u = 0$ then a possible orbit (such that $\dot{r} = \ddot{\theta} = 0$) is with $r = \rho$ and $\dot{\theta} = \omega = \sqrt{c/\rho^3}$.

Recall that one reason for taking an interest in linear models is that they tell us about controllability around an equilibrium point. Imagine there is a perturbing force. Take coordinates of perturbation

$$x_1 = r - \rho, \quad x_2 = \dot{r}, \quad x_3 = \theta - \omega t, \quad x_4 = \dot{\theta} - \omega.$$

Then, with $n = 4$, $m = 2$,

$$\dot{x} \sim \begin{pmatrix} 0 & 1 & 0 & 0 \\ 3\omega^2 & 0 & 0 & 2\omega\rho \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega/\rho & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1/\rho \end{pmatrix}\begin{pmatrix} u_r \\ u_\theta \end{pmatrix} = Ax + Bu\,.$$

It is easy to check that $M_2 = \begin{bmatrix} B & AB \end{bmatrix}$ has rank 4 and that therefore the system is controllable.

But suppose $u_\theta = 0$ (tangential thrust fails). Then

$$B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad M_4 = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & -\omega^2 \\ 1 & 0 & -\omega^2 & 0 \\ 0 & 0 & -2\omega/\rho & 0 \\ 0 & -2\omega/\rho & 0 & 2\omega^3/\rho \end{bmatrix}.$$

Since $(2\omega\rho, 0, 0, \rho^2)M_4 = 0$, this is singular and has rank 3. The uncontrollable component is the angular momentum, $2\omega\rho\delta r + \rho^2\delta\dot{\theta} = \delta(r^2\dot{\theta})|_{r=\rho,\dot{\theta}=\omega}$.

On the other hand, if $u_r = 0$ then the system is controllable. We can change the radius by tangential braking or thrust.