7 LQ Models

We present the LQ regulation model in discrete and continuous time, the Riccati equation, its validity in the model with additive white noise.

7.1 The LQ regulation model

The elements needed to define a control optimization problem are specification of (i) the dynamics of the process, (ii) which quantities are observable at a given time, and (iii) an optimization criterion.

In the **LQG model** the plant equation and observation relations are linear, the cost is quadratic, and the noise is Gaussian (jointly normal). The LQG model is important because it has a complete theory and introduces some key concepts, such as controllability, observability and the certainty-equivalence principle.

Begin with a model in which the state x_t is fully observable and there is no noise. The plant equation of the time-homogeneous $[A, B, \cdot\ ]$ system has the linear form

$$x_t = Ax_{t-1} + Bu_{t-1}\,, \tag{7.1}$$

where $x_t \in \mathbb{R}^n$, $u_t \in \mathbb{R}^m$, A is $n \times n$ and B is $n \times m$. The cost function is

$$\mathbf{C} = \sum_{t=0}^{h-1} c(x_t, u_t) + \mathbf{C}_h(x_h)\,, \tag{7.2}$$

with one-step and terminal costs

$$c(x, u) = x^\top Rx + u^\top Sx + x^\top S^\top u + u^\top Qu = \begin{bmatrix} x \\ u \end{bmatrix}^\top \begin{bmatrix} R & S^\top \\ S & Q \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}, \tag{7.3}$$

$$\mathbf{C}_h(x) = x^\top \Pi_h x\,. \tag{7.4}$$

All quadratic forms are non-negative definite, and Q is positive definite. There is no loss of generality in assuming that R, Q and Π_h are symmetric. This is a model for **regulation** of (x, u) to the point $(0, 0)$ (i.e., steering to a critical value).

To solve the optimality equation we shall need the following lemma.

Lemma 7.1 *Suppose x, u are vectors. Consider a quadratic form*

$$\begin{pmatrix} x \\ u \end{pmatrix}^\top \begin{pmatrix} \Pi_{xx} & \Pi_{xu} \\ \Pi_{ux} & \Pi_{uu} \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix}.$$

Assume it is symmetric and $\Pi_{uu} > 0$, i.e., positive definite. Then the minimum with respect to u is achieved at

$$u = -\Pi_{uu}^{-1}\Pi_{ux}x,$$

and is equal to

$$x^\top \left[\Pi_{xx} - \Pi_{xu}\Pi_{uu}^{-1}\Pi_{ux}\right] x.$$

Proof. Suppose the quadratic form is minimized at u. Then

$$\begin{pmatrix} x \\ u+h \end{pmatrix}^\top \begin{pmatrix} \Pi_{xx} & \Pi_{xu} \\ \Pi_{ux} & \Pi_{uu} \end{pmatrix} \begin{pmatrix} x \\ u+h \end{pmatrix}$$

$$= x^\top \Pi_{xx}x + 2x^\top \Pi_{xu}u + \underbrace{2h^\top \Pi_{ux}x + 2h^\top \Pi_{uu}u} + u^\top \Pi_{uu}u + h^\top \Pi_{uu}h\,.$$

To be stationary at u, the underbraced linear term in $h^\top$ must be zero, so

$$u = -\Pi_{uu}^{-1}\Pi_{ux}x\,,$$

and the optimal value is $x^\top \left[\Pi_{xx} - \Pi_{xu}\Pi_{uu}^{-1}\Pi_{ux}\right] x$. ∎

Theorem 7.2 *Assume the structure of (7.1)–(7.4). Then the value function has the quadratic form*

$$F(x, t) = x^\top \Pi_t x\,, \quad t < h\,, \tag{7.5}$$

and the optimal control has the linear form

$$u_t = K_t x_t\,, \quad t < h\,.$$

The time-dependent matrix Π_t satisfies the Riccati equation

$$\Pi_t = f\Pi_{t+1}\,, \quad t < h\,, \tag{7.6}$$

where f is an operator having the action

$$f\Pi = R + A^\top \Pi A - (S^\top + A^\top \Pi B)(Q + B^\top \Pi B)^{-1}(S + B^\top \Pi A)\,, \tag{7.7}$$

and Π_h has the value prescribed in (7.4). The $m \times n$ matrix K_t is given by

$$K_t = -(Q + B^\top \Pi_{t+1}B)^{-1}(S + B^\top \Pi_{t+1}A)\,, \quad t < h\,.$$

Proof. Assertion (7.5) is true at time h. Assume it is true at time $t + 1$. Then

$$F(x, t) = \inf_u \left[c(x, u) + (Ax + Bu)^\top \Pi_{t+1}(Ax + Bu)\right]$$

$$= \inf_u \left[\begin{pmatrix} x \\ u \end{pmatrix}^\top \begin{pmatrix} R + A^\top \Pi_{t+1}A & S^\top + A^\top \Pi_{t+1}B \\ S + B^\top \Pi_{t+1}A & Q + B^\top \Pi_{t+1}B \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix}\right]$$

By Lemma 7.1 the minimum is achieved by $u = K_t x$, and the form of f comes from this also. ∎

7.2 The Riccati recursion

The backward recursion (7.6)–(7.7) is called the **Riccati equation**. Note that

(i) S can be normalized to zero by choosing a new control $u^* = u + Q^{-1}Sx$, and setting $A^* = A - BQ^{-1}S$, $R^* = R - S^\top Q^{-1}S$.

(ii) The optimally controlled process obeys $x_{t+1} = \Gamma_t x_t$. Here Γ_t is called the **gain matrix** and is given by

$$\Gamma_t = A + BK_t = A - B(Q + B^\top \Pi_{t+1} B)^{-1}(S + B^\top \Pi_{t+1} A)\,.$$

(iii) An equivalent expression for the Riccati equation is

$$f\Pi = \inf_K \left[R + K^\top S + S^\top K + K^\top QK + (A+BK)^\top \Pi (A+BK)\right]\,.$$

(iv) We might have carried out exactly the same analysis for a time-heterogeneous model, in which the matrices A, B, Q, R, S are replaced by A_t, B_t, Q_t, R_t, S_t.

(v) We do not give details, but comment that it is possible to analyse models in which

$$x_{t+1} = Ax_t + Bu_t + \alpha_t\,,$$

for a known sequence of disturbances $\{\alpha_t\}$, or in which the cost function is

$$c(x,u) = \begin{bmatrix} x - \bar{x}_t \\ u - \bar{u}_t \end{bmatrix}^\top \begin{bmatrix} R & S^\top \\ S & Q \end{bmatrix} \begin{bmatrix} x - \bar{x}_t \\ u - \bar{u}_t \end{bmatrix}.$$

so that the aim is to track a sequence of values $(\bar{x}_t, \bar{u}_t)$, $t = 0, \dots, h-1$.

7.3 White noise disturbances

Suppose the plant equation (7.1) is now

$$x_{t+1} = Ax_t + Bu_t + \epsilon_t,$$

where $\epsilon_t \in \mathbb{R}^n$ is vector **white noise**, defined by the properties $E\epsilon = 0$, $E\epsilon_t\epsilon_t^\top = N$ and $E\epsilon_t\epsilon_s^\top = 0$, $t \neq s$. The DP equation is then

$$F(x,t) = \inf_u \Big[c(x,u) + E_\epsilon[(F(Ax + Bu + \epsilon, t+1)]\Big].$$

By definition $F(x,h) = x^\top \Pi_h x$. Try a solution $F(x,t) = x^\top \Pi_t x + \gamma_t$. This holds for $t = h$. Suppose it is true for $t+1$, then

$$\begin{aligned} F(x,t) &= \inf_u \left[c(x,u) + E(Ax + Bu + \epsilon)^\top \Pi_{t+1}(Ax + Bu + \epsilon) + \gamma_{t+1}\right] \\ &= \inf_u \left[c(x,u) + E(Ax + Bu)^\top \Pi_{t+1}(Ax + Bu)\right] \\ &\qquad + 2E\left[\epsilon^\top (Ax + Bu)\right] + E\left[\epsilon^\top \Pi_{t+1}\epsilon\right] + \gamma_{t+1} \\ &= \inf_u [\cdots] + 0 + \operatorname{tr}(N\Pi_{t+1}) + \gamma_{t+1}\,. \end{aligned}$$

Here we use the fact that

$$E\left[\epsilon^\top \Pi \epsilon\right] = E\left[\sum_{ij} \epsilon_i \Pi_{ij} \epsilon_j\right] = E\left[\sum_{ij} \epsilon_j \epsilon_i \Pi_{ij}\right] = \sum_{ij} N_{ji}\Pi_{ij} = \operatorname{tr}(N\Pi)\,.$$

Thus (i) Π_t follows the same Riccati equation as before, (ii) the optimal control is $u_t = K_t x_t$, and (iii)

$$F(x,t) = x^\top \Pi_t x + \gamma_t = x^\top \Pi_t x + \sum_{j=t+1}^{h} \operatorname{tr}(N\Pi_j)\,.$$

The final term can be viewed as the cost of correcting future noise. In the infinite horizon limit of $\Pi_t \to \Pi$ as $t \to \infty$, we incur an average cost per unit time of $\operatorname{tr}(N\Pi)$, and a transient cost of $x^\top \Pi x$ that is due to correcting the initial x.

7.4 LQ regulation in continuous-time

In continuous-time we take $\dot{x} = Ax + Bu$ and

$$\mathbf{C} = \int_0^h \begin{pmatrix} x \\ u \end{pmatrix}^\top \begin{pmatrix} R & S^\top \\ S & Q \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} dt + (x^\top \Pi x)_h\,.$$

We can obtain the continuous-time solution from the discrete time solution by moving forward in time in increments of Δ. Make the following replacements.

$$x_{t+1} \to x_{t+\Delta}, \quad A \to I + A\Delta, \quad B \to B\Delta, \quad R,\, S,\, Q \to R\Delta,\, S\Delta,\, Q\Delta\,.$$

Then as before, $F(x,t) = x^\top \Pi x$, where Π obeys the Riccati equation

$$\frac{\partial \Pi}{\partial t} + R + A^\top \Pi + \Pi A - (S^\top + \Pi B)Q^{-1}(S + B^\top \Pi) = 0\,.$$

This is simpler than the discrete time version. The optimal control is

$$u(t) = K(t)x(t)$$

where

$$K(t) = -Q^{-1}(S + B^\top \Pi)\,.$$

The optimally controlled plant equation is $\dot{x} = \Gamma(t)x$, where

$$\Gamma(t) = A + BK = A - BQ^{-1}(S + B^\top \Pi)\,.$$