Mathematics for Operations Research Examples 1

1. **SOLUTION**. The Lagrangian is

$$L(x,\lambda) = -2x_1^2 - x_2^2 + x_1x_2 + 8x_1 + 3x_2 - \lambda(3x_1 + x_2 - 10)\,.$$

This has a stationary point where

$$-4x_1 + x_2 + 8 - 3\lambda = 0 \quad \text{and} \quad -2x_2 + x_1 + 3 - \lambda = 0$$

The solution is

$$x_1 = \frac{19-7\lambda}{7} \quad \text{and} \quad x_2 = \frac{20-7\lambda}{7}\,.$$

Now choose λ so that

$$3x_1 + x_2 = 3\left(\frac{19-7\lambda}{7}\right) + \frac{20-7\lambda}{7} = 10\,.$$

This gives $\lambda = 1/4$ and $(x_1, x_2) = (69/28, 73/28)$. The second derivative matrix is

$$\left(\frac{\partial L^2}{\partial x_i \partial x_j}\right) = \begin{pmatrix} -4 & 1 \\ 1 & -2 \end{pmatrix}$$

which is nonnegative definite, proving that the stationary point is a maximum.

2. **SOLUTION**.

	x_1	x_2	x_3	z_1	z_2	z_3	a	
z_1	1	1	1	1	0	0	0	2
z_2	2	1	0	0	1	0	0	3
a	2	1	$\boxed{3}$	0	0	−1	1	3
Phase I	0	0	0	0	0	0	1	0
	−2	−1	−3	0	0	1	0	−3
Phase II	4	4	1	0	0	0	0	0

It is the second Phase I row that we use, where the objective function of a has been written in terms on the nonbasic variables. Pivoting on the entry shown, we have

	x_1	x_2	x_3	z_1	z_2	z_3	a	
z_1	1/3	2/3	0	1	0	1/3	−1/3	1
z_2	2	1	0	0	1	0	0	3
a	2/3	1/3	1	0	0	−1/3	1/3	1
Phase I	0	0	0	0	0	0	1	0
Phase II	10/3	11/3	1	0	0	1/3	−1/3	−1

At this point, a has been minimized to 0. The bottom row is already nonnegative, so no further interations are needed in Phase II. The optimal solution is $(x, z) = (0, 0, 1, 1, 0, 3)$.

3. **SOLUTION**.

(a) is true if the problem is nondegenerate. In that case the only way a variable can leave the basis is if we pivot in its column. The entry that is in the bottom row of the pivot column is always strictly positive. So when a basic variable leaves the basis the value of the entry that appears in the bottom row of that variable's column goes from being 0 to being strictly negative. So this column cannot immediately be the next pivot column.

However, if we have a degenerate problem such as maximize $x_1 + 10x_2$ subject to $x_1 + x_3 = 1$, $3x_2 + x_1 + x_4 = 4$ and $2x_2 + x_1 + x_5 = 3$ we might go from $(0, 0, 1, 4, 3)$ to $(1, 0, 0, 3, 2)$ to $(1, 1, 0, 0, 0)$ to $(0, 4/3, 1, 0, 1/3)$. Note that the variable x_5 leaves the basis and then immediately reenters.

(b) is false. Consider the problem maximize $x_2 + 2x_3$ subject to $x_1 + x_2 + x_3 = 1$. We might start at $x = (1, 0, 0)$, move to $(0, 1, 0)$ and then to $(0, 0, 1)$. In this example, x_2 enters the basis and then immediately leaves.

4. **SOLUTION**. We take the Phase I objective function: minimize $\sum y_i$. Suppose y_1 is the first artificial variable to leave. At that point, let us reset the Phase I objective function to $M_1y_1 + y_2 + \cdots + y_\ell$. Then the Phase I objective function row gets an additional M_1 added to the entry in the y_1 column. We imagine taking M_1 so large that, as further iterations proceed, this entry of the tableau never again becomes negative. Recall that we only pivot in columns where the entry in the Phase I objective row is negative. Hence y_1 never reenters the basis. Note that if we can minimize $\sum y_i$ to 0 then we can also minimize $M_1y_1 + y_2 + \cdots + y_\ell$ to 0.

5. **SOLUTION**. Note that D is certainly feasible, since $y = 0$ is a feasible solution. If D is bounded then D can be solved with the simplex algorithm (or by Lagrangian) methods; either of these implies that there exists by construction (or by existence of the Lagrangian multipliers) an optimal (and feasible) solution to P. If D is unbounded then there exists y such that $y^\top b = -1$ and $y^\top A \geq 0$. But if there were a feasible x for P such that $Ax = b$ and $x \geq 0$ we would have $-1 = y^\top b = y^\top(Ax) = (y^\top A)x \geq 0$. Hence P cannot be feasible.

Farkas lemma is simply the statement that either D is unbounded, i.e., (b), or D is bounded, which is if and only if P is feasible, i.e. (a).

6. **SOLUTION**.

x_1	x_2	z_1	z_2	z_3	
−3	$\boxed{4}$	1	0	0	4
3	2	0	1	0	11
2	−1	0	0	1	5
1	2	0	0	0	0

x_1	x_2	z_1	z_2	z_3	
$-3/4$	1	$1/4$	0	0	1
$\boxed{9/2}$	0	$-1/2$	1	0	9
$5/4$	0	$1/4$	0	1	6
$5/2$	0	$-1/2$	0	0	-2

x_1	x_2	z_1	z_2	z_3	
0	1	$1/6$	$1/6$	0	$5/2$
1	0	$-1/9$	$2/9$	0	2
0	0	$7/18$	$-5/18$	1	$7/2$
0	0	$-2/9$	$-5/9$	0	-7

We now have the optimal solution, but it is not in integers. From the first row we have the constraint

$$x_2 + \tfrac{1}{6}z_1 + \tfrac{1}{6}z_2 = \tfrac{5}{2}$$

This implies that an integer solution must satisfy $x_2 \le 2$. We add this constraint to get

x_1	x_2	z_1	z_2	z_3	z_4	
0	1	$1/6$	$1/6$	0	0	$5/2$
1	0	$-1/9$	$2/9$	0	0	2
0	0	$7/18$	$-5/18$	1	0	$7/2$
0	1	0	0	0	1	2
0	0	$\boxed{-1/6}$	$-1/6$	0	1	$-1/2$
0	0	$-2/9$	$-5/9$	0	0	-7

x_1	x_2	z_1	z_2	z_3	z_4	
0	1	0	0	0	1	2
1	0	0	$1/3$	0	$-2/3$	$7/3$
0	0	0	$-2/3$	1	$7/3$	$7/3$
0	0	1	1	0	-6	3
0	0	0	$-1/3$	0	$-4/3$	$-19/3$

The solution is not yet in integers. So we can use the constaint from the second row

$$x_1 + \tfrac{1}{3}z_2 - \tfrac{2}{3}z_4 \le \tfrac{7}{3}$$

to get $x_1 - z_4 \le 2$.

x_1	x_2	z_1	z_2	z_3	z_4	z_5	
0	1	0	0	0	1	0	2
1	0	0	$1/3$	0	$-2/3$	0	$7/3$
0	0	0	$-2/3$	1	$7/3$	0	$7/3$
0	0	1	1	0	-6	0	3
1	0	0	0	0	-1	1	2
0	0	0	$\boxed{-1/3}$	0	$-1/3$	1	$-1/3$
0	0	0	$-1/3$	0	$-4/3$	0	$-19/3$

x_1	x_2	z_1	z_2	z_3	z_4	z_5	
0	1	0	0	0	1	0	2
1	0	0	0	0	-1	1	2
0	0	0	0	1	3	-2	3
0	0	1	0	0	-7	3	2
0	0	0	1	0	1	-3	1
0	0	0	0	0	-1	-1	-6

So the solution is $x_1 = x_2 = 2$, with an optimum value of 6.

7. **SOLUTION**. Define $x_i = 0, 1$ as X_i is true or false. Let $\bar{x}_i = 1 - x_i$. Then the satisfiability question is the same as asking whether there is a feasible solution to the 0–1 linear programming problem

$$\begin{aligned}
&\text{maximize } 0 \text{ subject to}\\
&x_i, \bar{x}_i \in \{0,1\}, \quad x_i + \bar{x}_i = 1, \quad i = 1, \dots, 6\\
&x_1 + \bar{x}_2 + x_6 \ge 1\\
&\bar{x}_2 + \bar{x}_4 \ge 1\\
&x_3 + x_5 + x_6 \ge 1
\end{aligned}$$

Now satifiability is in $\mathcal{NP}$ because given a certificate (which is just an assignment of truth values to the $X_1, X_2, X_3, X_4, X_5, X_6$) we can check the truth of a clause in a time that is linear in the size of the clause. Clearly, satifiability reduces to an instance of 0–1 linear programming. Hence 0–1 linear programming is $\mathcal{NP}$-complete.

8. **SOLUTION**. If G has a clique of size k, then in G' these k nodes are such that no pair of them has a joining edge. This means that all edges of G' must have an endpoint amongst the other $n - k$ nodes, i.e., a vertex cover of size $n - k$.

   Similarly, if G' has a vertex cover of size $n - k$, then amongst the other k nodes there cannot be any joining edges. Thus in G there are edges between every pair of these nodes and they form a clique of size k.

   Vertex Cover is in $\mathcal{NP}$ since if there exists a vertex cover of size $n - k$ then a certificate for it is a list of the $n - k$ nodes in the cover. We must simply check that every edge ends in at least one node of this set. Since there are at most n^2 edges we can check this fact in time $O(n^2(n-k))$ time. Now **Clique** reduces to **Vertex Cover** (by the result in the first two paragraphs), since calculation of G' from G is a polynomial time calculation, in fact, $O(n^2)$. Thus **Vertex Cover** is at least as hard as **Clique**, which proves that **Vertex Cover** is $\mathcal{NP}$-complete.

9. **SOLUTION**.

   (a) If (z_1, z_2) is an edge then there is a clause equivalent to $(\bar{z}_1 \vee z_2)$. This is equivalent to $(z_2 \vee \bar{z}_1)$. So there must be an edge $(\bar{z}_2, \bar{z}_1)$.

   (b) If there is an edge (z_1, z_2) then there is a clause equivalent to $(\bar{z}_1 \vee z_2)$. So $z_1 =$T $\implies$ $\bar{z}_1 =$F and so we must have $z_2 =$T.

   (c) If there is a z such that (i) is true then there is a path from z to $\bar{z}$, so z cannot be true, by (b). Similarly, the existence of a path from $\bar{z}$ to z means that $\bar{z}$ cannot be true. So the instance is unsatisfiable.

   Now suppose that (i) is not true. Pick an unassigned variable x in some clause. Assign $x =$T (making that clause true), and also assign T to all vertices that can be reached along paths from x. Assign F to the negations of all variables so assigned. Repeat until all variables are assigned. The algorithm is well-defined in the sense that it never sets both a variable and its negation T, because if there were a paths $x \to y$ and $x \to \bar{y}$ then there would be a path $y \to \bar{x}$, which means there is a path $x \to \bar{x}$ and so x cannot have been assigned T, by (b).

   Notice that we cannot end up with some clause $(x \vee y)$ in which $x = y =$F. If there were such a clause then there exists the edge $(\bar{x}, y)$ and $\bar{x} =$T. Also there exists the edge $(\bar{y}, x)$ and $\bar{y}$=T. So at whatever step of the algorithm we first set $\bar{x} =$T or $\bar{y} =$T, that must also have set $y =$T or $x =$T, respectively.

It follows that 2-SAT$\in$ $\mathcal{P}$ since we can determine whether or not (2) is true in polynomial time.

10. **SOLUTION**.

   (a) If there is a clique of size m we have found m variables one from each clause no one of which is the complement of the other. By setting these variables true we have satisfied all the clauses. Conversely, if the clauses are satisfiable then one variable in each clause is true and none of these is the complement of the other. Thus there are edges between the corresponding m vertices in G.

   (b) The mapping of our instance of 3-SAT to the graph G can clearly be done in polynomial time. It shows that 3-SAT is no harder than CDP. But 3-SAT is $\mathcal{NP}$-complete. So CDP must be $\mathcal{NP}$-complete. If there were a polynomial time algorithm for CDP then there would be one for 3-SAT.

11. **SOLUTION**. The Hirsch conjecture is easy to prove when $d = 2$. We simply need to check that two vertices of a polygon of n sides are never further apart than a walk along $n-2$ edges. In fact, there walk need never be more than $\lfloor n/2 \rfloor$ edges.

12. **SOLUTION**. This question is straightforward and is really just to get you thinking. The only slightly tricky bit is at the end. We might worry that if $(1, 2, 3, 4)$ is the optimum, and $(2, 1, 4, 3)$ is second-best then we cannot get from $(2, 1, 3, 4)$ to $(1, 2, 3, 4)$ is one step (since $(2, 1, 4, 3) = \{(1, 2), (3, 4)\}$ is two cycles from $(1, 2, 3, 4)$). However, $(2, 1, 3, 4)$ cannot be second best, since both $(1, 2, 4, 3)$ and $(2, 1, 3, 4)$ must be better than $(2, 1, 4, 3)$.

    The same is true for $n = 5$ since every permuation on $\{1, 2, 3, 4, 5\}$ can be written as the product of no more than 2 disjoint cycles.

    The polytope of the transportation problem has also been studied. This has

$$\sum_{j=1}^{m} x_{ij} = s_i\,, \quad \text{for all } i\,; \quad \sum_{i=1}^{n} x_{ij} = d_j\,, \quad \text{for all } j\,; \quad x_{ij} \in \{0, 1\}\,, \quad \text{for all } i, j.$$

    It has been proved that the diameter is no more than $8(m + n - 1)$. If the Hirsch conjecture is true then the diameter is no greater than $m + n - 1$.

    The monotone Hirsch conjecture has been shown to be false in general. Howver, might it be true for the transportation polytope? The general 'strict monotone Hirsch conjecture' is still open. This says that it is possible to find a monotone nondecreasing path from the vertex where $c^\top x$ is minimized to the vertex where it is maximized that is no longer than $n - d$ steps.

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