Symmetric Rendezvous Search

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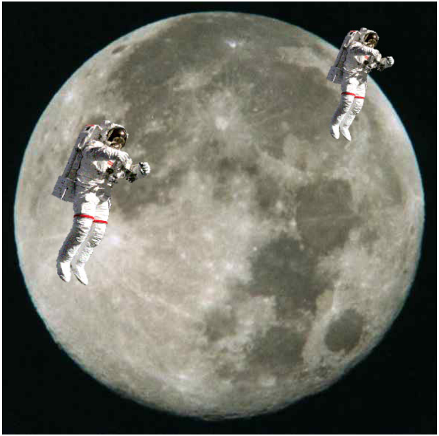
R. R. Weber, Optimal symmetric rendezvous search on three locations, Math Oper Res., 37(1): 111-122, 2012.

Rendezvous search on a sphere

Alpern (1976) proposed the following problem.



Two astronauts land at random spots on a planet (which is assumed to be a uniform sphere, without any known distinguishing marks or directions) How should they move so as to be within 1 kilometre of one another in the least expected time?

Mozart cafe problem

Two friends travelling independently to Vienna wish to meet for a coffee on the afternoon they arrive. Neither has been to Vienna before, but they guess it must have a Mozart Cafe. So in an email exchange they agree to meet at the Mozart Cafe.

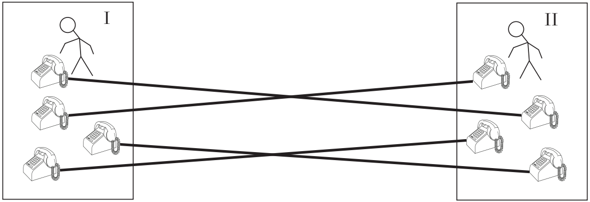
Unfortunately, upon arrival they each discover that there are in fact $m > 1$ Mozart Cafes in Vienna.

Assuming they have no way to communicate, what should they do?

Telephone coordination game

In each of two rooms there is a player and n telephones.

Phones are connected pairwise in some unknown fashion.



At attempts $1, 2, \ldots$, the players pick up phones and say "hello".

Their common aim is to minimize the expected number of attempts until they hear one another.

Symmetric rendezvous search on n locations

Assumptions

1. Two players are randomly placed at two distinct on n vertices of complete graph K_n.
2. No common labelling of the locations.
3. At steps $1, 2, \ldots$, each player visits a location.
4. Players adopt identical (randomizing) strategies.

What should their common strategy be if they wish to meet in the least expected number of steps?

Some possible strategies

Move-at-random If at each discrete step $1, 2, \ldots$ each player were to locate himself at a randomly chosen location, then the expected time to meet would be n. E.g.,

$$ET = 1 + \frac{n-1}{n}ET \implies ET = n\,.$$

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Wait-for-mommy Suppose the players could break symmetry (or had some prior agreement). Now it is best for one player to remain stationary while the other tours all other locations in random order. They will meet (on average) half way through the tour. So

$$ET = \tfrac{1}{n-1}\big(1 + 2 + \cdots + (n-1)\big) = \tfrac{1}{2}n\,.$$

Wait-for-mommy

"*According to a recent National Geographic Video, the mother kangaroo teaches its baby to find the nearest bush and hide (wait) when the two become separated.*" (Alpern)

E.J. Anderson and R.R. Weber. The rendezvous problem on discrete locations. *J. Appl. Prob.* 27, 839-851, 1990.

Theorem 1 *In the asymmetric rendezvous search game on n locations the optimal strategy is wait-for-mommy. (Anderson-Weber, 1990)*

The Anderson-Weber strategy on K_3

On K_3, **AW** specifies that in each block of two consecutive steps, each player should, independently of the other, either stay at his initial location or tour the other two locations in random order, doing these with respective probabilities $p = \frac{1}{3}$ and $1 - p = \frac{2}{3}$.

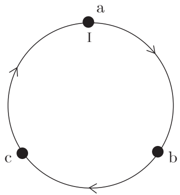
AW gives $ET = \frac{5}{2}$, whereas *move-at-random* gives $ET = 3$.

Theorem 2 *On K_3,* **AW** *minimizes ET.*

Corollary. $w = \frac{5}{2}$ *on K_3.*

Formulation of the problem

Suppose the three locations are arranged around a circle.



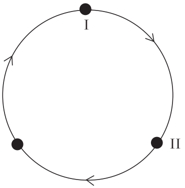
Each player calls his home location 'a'.

Each player chooses a direction he calls 'clockwise' and the labels that are one and two locations clockwise of home as 'b' and 'c' respectively.

A sequence of a player's moves can now be described.
E.g., a player's first 6 moves might be '$ababbc$'.

Make the problem easier by providing the players with a common notion of clockwise. (We'll see this does not actually help.)

Player II starts one position clockwise of Player I.



$$B_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

Matrix B_1 has '1' if after the first step they do not meet, and '0' if they do.

Rows of B_1 correspond to I playing a, b or c.

Columns of B_1 correspond to II playing a, b or c.

The minimum of $P(T > 2)$

The indicator matrix for not meeting within 2 steps is

$$B_2 := B_1 \otimes B_1 = \begin{pmatrix} B_1 & B_1 & 0 \\ 0 & B_1 & B_1 \\ B_1 & 0 & B_1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

Rows 1–9 (and columns 1–9) correspond respectively to Player I (or II) playing patterns of moves over the first two steps of $aa, ab, ac, ba, bb, bc, ca, cb, cc$.

$ET = \sum_{k=0}^{\infty} P(T > k)$.

AW minimizes $P(T > 2)$

Let $\bar{B}_2 = \frac{1}{2}(B_2 + B_2^\top)$ (to account for II starting either one or two locations clockwise of I).

$$P(T>2) = p^\top \bar{B}_2 p = p^\top \begin{pmatrix} 1 & \frac12 & \frac12 & \frac12 & \frac12 & 0 & \frac12 & 0 & \frac12 \\ \frac12 & 1 & \frac12 & 0 & \frac12 & \frac12 & \frac12 & \frac12 & 0 \\ \frac12 & \frac12 & 1 & \frac12 & 0 & \frac12 & 0 & \frac12 & \frac12 \\ \frac12 & 0 & \frac12 & 1 & \frac12 & \frac12 & \frac12 & \frac12 & 0 \\ \frac12 & \frac12 & 0 & \frac12 & 1 & \frac12 & 0 & \frac12 & \frac12 \\ 0 & \frac12 & \frac12 & \frac12 & \frac12 & 1 & \frac12 & 0 & \frac12 \\ \frac12 & \frac12 & 0 & \frac12 & 0 & \frac12 & 1 & \frac12 & \frac12 \\ 0 & \frac12 & \frac12 & \frac12 & \frac12 & 0 & \frac12 & 1 & \frac12 \\ \frac12 & 0 & \frac12 & 0 & \frac12 & \frac12 & \frac12 & \frac12 & 1 \end{pmatrix} p$$

Minimizing $p^\top \bar{B}_2 p$ is a difficult quadratic programming problem.

Minimizer is $p^\top = \frac{1}{3}(1,0,0,0,0,1,0,1,0)$, where '$aa$', '$bc$' and '$cb$' are to be chosen equally likely, (which is **AW**).

Another minimizer is $p^\top = (0,1,0,1,0,0,0,0,1)$, where '$ab$', '$ba$' and '$cc$' are to be chosen equally likely.

A quadratic programming problem

Hard to minimize $p^\top \bar{B}_2 p$ because $\bar{B}_2$ is not positive semidefinite. It's eigenvalues are $\{4, 1, 1, 1, 1, 1, 1, -\frac{1}{2}, -\frac{1}{2}\}$.

This means that there can be local minima to $p^\top \bar{B}_2 p$.

E.g., $p = \frac{1}{9}(1, 1, 1, 1, 1, 1, 1, 1, 1)$, is a local minimum; with $p^\top \bar{B}_2 p = \frac{4}{9}$. Not a global minimum.

In general, if a matrix C is not positive semidefinite, the following problem is NP-hard:

$$\text{minimize } \; p^\top C p \; : \; p \geq 0 \, , \; 1^\top p = 1 \, .$$

A method for finding lower bounds

We can obtain a lower bound on the solution as follows.

$$\begin{aligned}&\min\{p^\top C p \,:\, p \geq 0\,,\ 1^\top p = 1\}\\ &\qquad = \min\{\operatorname{trace}(Cpp^\top) \,:\, p \geq 0\,,\ 1^\top p = 1\}\\ &\qquad \geq \min\{\operatorname{trace}(CX) \,:\, X \succeq 0\,,\ X \geq 0\,,\ \operatorname{trace}(JX) = 1\}\,,\end{aligned}$$

where $J = 11^\top$ is a matrix of all 1s.

This is by using the fact that if p satisfies the l.h.s. constraints, then $X = pp^\top$ satisfies the r.h.s. constraints.

Semidefinite programming problems

'linear programming for the 21st century'.

Given symmetric matrices $C, A_1, \ldots, A_m$, consider the problem

$$\text{minimize}\,\{\text{trace}(CX) \\ : \; X \succeq 0\,,\; X \geq 0\,,\; \text{trace}(A_i X) = b_i,\; i = 1, \ldots, m\}\,.$$

This is a *Semidefinite Programming Problem* (SDP).

The minimization is over the components of X.
This can mean lots of decision variables.
If X is $j \times j$ and symmetric, then there are $j(j-1)/2$ variables.

SDPs can be solved to any degree of numerical accuracy using interior point algorithms (e.g., using Matlab and sedumi).

A lower bound on $p^\top \bar{B}_2 p$

As a relaxation of the quadratic program:

$$\text{minimize}\,\{p^\top \bar{B}_2 p \,:\, p \geq 0\,,\; 1^\top p = 1\}\,,$$

we consider the SDP:

$$\text{minimize}\,\{\text{trace}(\bar{B}_2 X) \,:\, X \succeq 0\,,\; X \geq 0\,,\; \text{trace}(J_2 X) = 1\}\,,$$

where J_2 is the 9×9 matrix of 1s. There are 36 decision variables. We find that the minimum value is $1/3$.

But $p^\top \bar{B}_2 p = 1/3$ for $p^\top = \frac{1}{3}(1, 0, 0, 0, 0, 1, 0, 1, 0)$.

So we may conclude that $1/3$ is the minimal value of $p^\top \bar{B}_2 p$.

Lower bounds on w_k

Solving SDPs, we get lower bounds on
$w_k = \min \; E[\min\{T, k+1\}]$.

Lower bounds when players have a common clockwise:

k	1	2	3	4
w_k	$\frac{5}{3}$	2	$\frac{20}{9}$	$\frac{21}{9}$

Lower bounds when players have no common clockwise:

k	1	2	3	4	5
w_k	$\frac{5}{3}$	2	$\frac{20}{9}$	$\frac{21}{9}$	$\frac{65}{27}$ ‡

Observations

1. These lower bounds so prove that **AW** minimizes $E[\min\{T, k+1\}]$ as far as $k = 4$.
2. However it is computationally infeasible to go much further. The number of decision variables in the SDP is 3240 when $k = 4$. For $k = 5$ it would be 29403.

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The optimality of **AW** for K_3

Theorem 3 *The* **AW** *strategy is optimal for the symmetric rendezvous search game on* K_3*, minimizing* $E[\min\{T, k+1\}]$ *to* w_k *for all* $k = 1, 2, \dots$ *, where*

$$w_k = \begin{cases} \frac{5}{2} - \frac{5}{2} 3^{-\frac{k+1}{2}}, & \text{when } k \text{ is odd,} \\ \frac{5}{2} - \frac{3}{2} 3^{-\frac{k}{2}}, & \text{when } k \text{ is even.} \end{cases}$$

Consequently, the minimal achievable value of ET *is* $w = \frac{5}{2}$.

$\{w_k\}_0^\infty = \{1, \frac{5}{3}, 2, \frac{20}{9}, \frac{21}{9}, \frac{65}{27}, \dots\}$.

The minimum of $E[\min\{T, 3\}]$

We can take $p^\top = \frac{1}{3}(1,0,0,0,0,1,0,1,0)$ and

$$M_2 = \begin{pmatrix} 3 & 3 & 2 & 3 & 3 & 2 & 1 & 1 & 1 \\ 2 & 3 & 3 & 2 & 3 & 3 & 1 & 1 & 1 \\ 3 & 2 & 3 & 3 & 2 & 3 & 1 & 1 & 1 \\ 1 & 1 & 1 & 3 & 3 & 2 & 3 & 3 & 2 \\ 1 & 1 & 1 & 2 & 3 & 3 & 2 & 3 & 3 \\ 1 & 1 & 1 & 3 & 2 & 3 & 3 & 2 & 3 \\ 3 & 3 & 2 & 1 & 1 & 1 & 3 & 3 & 2 \\ 2 & 3 & 3 & 1 & 1 & 1 & 2 & 3 & 3 \\ 3 & 2 & 3 & 1 & 1 & 1 & 3 & 2 & 3 \end{pmatrix}$$

$$\geq H_2 = \begin{pmatrix} 3 & 3 & 2 & 3 & 3 & 2 & 1 & 1 & 0 \\ 2 & 3 & 3 & 2 & 3 & 3 & 0 & 1 & 1 \\ 3 & 2 & 3 & 3 & 2 & 3 & 1 & 0 & 1 \\ 1 & 1 & 0 & 3 & 3 & 2 & 3 & 3 & 2 \\ 0 & 1 & 1 & 2 & 3 & 3 & 2 & 3 & 3 \\ 1 & 0 & 1 & 3 & 2 & 3 & 3 & 2 & 3 \\ 3 & 3 & 2 & 1 & 1 & 0 & 3 & 3 & 2 \\ 2 & 3 & 3 & 0 & 1 & 1 & 2 & 3 & 3 \\ 3 & 2 & 3 & 1 & 0 & 1 & 3 & 2 & 3 \end{pmatrix}.$$

Eigenvalues of $\bar{M}_2$ are $\{19, \frac{5}{2}, \frac{5}{2}, 1, 1, 1, 1, -\frac{1}{2}, -\frac{1}{2}\}$, so it is not positive semidefinite.

Eigenvalues of $\bar{H}_2$ are $\{18, 3, 3, \frac{3}{2}, \frac{3}{2}, 0, 0, 0, 0\}$ so $\bar{H}_2 \succeq 0$. Here

$$\bar{H}_2 p = \begin{pmatrix} 3 & \frac{5}{2} & \frac{5}{2} & 2 & \frac{3}{2} & \frac{3}{2} & 2 & \frac{3}{2} & \frac{3}{2} \\ \frac{5}{2} & 3 & \frac{5}{2} & \frac{3}{2} & 2 & \frac{3}{2} & \frac{3}{2} & 2 & \frac{3}{2} \\ \frac{5}{2} & \frac{5}{2} & 3 & \frac{3}{2} & \frac{3}{2} & 2 & \frac{3}{2} & \frac{3}{2} & 2 \\ 2 & \frac{3}{2} & \frac{3}{2} & 3 & \frac{5}{2} & \frac{5}{2} & 2 & \frac{3}{2} & \frac{3}{2} \\ \frac{3}{2} & 2 & \frac{3}{2} & \frac{5}{2} & 3 & \frac{5}{2} & \frac{3}{2} & 2 & \frac{3}{2} \\ \frac{3}{2} & \frac{3}{2} & 2 & \frac{5}{2} & \frac{5}{2} & 3 & \frac{3}{2} & \frac{3}{2} & 2 \\ 2 & \frac{3}{2} & \frac{3}{2} & 2 & \frac{3}{2} & \frac{3}{2} & 3 & \frac{5}{2} & \frac{5}{2} \\ \frac{3}{2} & 2 & \frac{3}{2} & \frac{3}{2} & 2 & \frac{3}{2} & \frac{5}{2} & 3 & \frac{5}{2} \\ \frac{3}{2} & \frac{3}{2} & 2 & \frac{3}{2} & \frac{3}{2} & 2 & \frac{5}{2} & \frac{5}{2} & 3 \end{pmatrix} \begin{pmatrix} \frac{1}{3} \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{3} \\ 0 \\ \frac{1}{3} \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \end{pmatrix}.$$

Thus p satisfies a Kuhn-Tucker condition for there to be a local minimum of $p^\top \bar{H}_2 p = 2$.

Since $\bar{H}_2 \succeq 0$, a local minimum is also a global minimum.

So $w_2 = 2$. This is achieved by **AW**.

Minimizing $E[\min\{T, k+1\}]$

Similarly, consider the problem of minimizing $E[\min\{T, k+1\}]$.

This is equivalent to minimizing $p^\top \bar{M}_k p$, where

$$M_k = J_k + B_1 \otimes J_{k-1} + \cdots + B_k \,.$$

As we did with H_2 for M_2, we look for H_k, such that $H_k \leq M_k$ and $\bar{H}_k \succeq 0$. This is a semidefinite programming problem

$$\text{maximize}\{\text{trace}(J_k H_k) \;:\; H_k \leq M_k,\; \bar{H}_k \succeq 0\}\,.$$

How can we find H_k?

$$\text{maximize}\{\text{trace}(J_2 H_2) \,:\, H_2 \leq M_2,\ \bar{H}_2 \succeq 0\}\,.$$

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$$H_2 = \begin{pmatrix}
3.0000 & 2.7951 & 1.8324 & 2.8005 & 2.8005 & 2.0000 & 0.8857 & 1.0000 & 0.8857 \\
1.8324 & 3.0000 & 2.7951 & 2.0000 & 2.8005 & 2.8005 & 0.8857 & 0.8857 & 1.0000 \\
2.7951 & 1.8324 & 3.0000 & 2.8005 & 2.0000 & 2.8005 & 1.0000 & 0.8857 & 0.8857 \\
0.8857 & 1.0000 & 0.8857 & 3.0000 & 2.7951 & 1.8324 & 2.8005 & 2.8005 & 2.0000 \\
0.8857 & 0.8857 & 1.0000 & 1.8324 & 3.0000 & 2.7951 & 2.0000 & 2.8005 & 2.8005 \\
1.0000 & 0.8857 & 0.8857 & 2.7951 & 1.8324 & 3.0000 & 2.8005 & 2.0000 & 2.8005 \\
2.8005 & 2.8005 & 2.0000 & 0.8857 & 1.0000 & 0.8857 & 3.0000 & 2.7951 & 1.8324 \\
2.0000 & 2.8005 & 2.8005 & 0.8857 & 0.8857 & 1.0000 & 1.8324 & 3.0000 & 2.7951 \\
2.8005 & 2.0000 & 2.8005 & 1.0000 & 0.8857 & 0.8857 & 2.7951 & 1.8324 & 3.0000
\end{pmatrix}$$

and $\min_p\{p^\top H_2 p\} = 1.9999889.$

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and $\min_p\{p^\top H_2 p\} = 1.9999889$. But $\min_p\{p^\top H_2 p\} = 2$ using

$$H_2 = \begin{pmatrix} 3 & 3 & 2 & 3 & 3 & 2 & 1 & 1 & 0 \\ 2 & 3 & 3 & 2 & 3 & 3 & 0 & 1 & 1 \\ 3 & 2 & 3 & 3 & 2 & 3 & 1 & 0 & 1 \\ 1 & 1 & 0 & 3 & 3 & 2 & 3 & 3 & 2 \\ 0 & 1 & 1 & 2 & 3 & 3 & 2 & 3 & 3 \\ 1 & 0 & 1 & 3 & 2 & 3 & 3 & 2 & 3 \\ 3 & 3 & 2 & 1 & 1 & 0 & 3 & 3 & 2 \\ 2 & 3 & 3 & 0 & 1 & 1 & 2 & 3 & 3 \\ 3 & 2 & 3 & 1 & 0 & 1 & 3 & 2 & 3 \end{pmatrix}.$$

How to construct H_k

Let us search for H_k of a special form. For $i = 0, \ldots, 3^k - 1$ we write $i_{\text{base } 3} = i_1 \cdots i_k$ (keeping k digits, including leading 0s); so $i_1, \ldots, i_k \in \{0, 1, 2\}$. Define

$$P_i = P_{i_1 \cdots i_k} = P_1^{i_1} \otimes \cdots \otimes P_1^{i_k} \, ,$$

where

$$P_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} .$$

Observe that $M_k = \sum_i m_k(i) P_i$, where m_k is the first row of M_k. This motivates seeking H_k of the form

$$H_k = \sum_{i=0}^{3^k-1} x_k(i) P_i \, .$$

Concluding steps of the proof

We want

1. $M_k = \sum_i m_k(i) P_i \geq H_k = \sum_i x_k(i) P_i$.
2. $\bar{H}_k \succeq 0$.

Since $P_0, \ldots, P_{3^k-1}$ commute they have common eigenvectors. Let $\omega = -\frac{1}{2} + i\frac{1}{2}\sqrt{3}$, a cube root of 1. Let $V_k = U_k + iW_k$.

$$V_k = V_1 \otimes V_{k-1}\,, \quad \text{where } V_1 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} .$$

Columns of V_k are eigenvectors of the P_i and also of M_k.
Columns of U_k are eigenvectors of the $\bar{P}_i$ and also of $\bar{M}_k$.

Our SDP becomes equivalent to a LP, with constraints
1. $m_k \geq x_k$ and 2. $U_k x_k \geq 0$.

We show that we may take $H_k = \sum_i x_k(i) P_i$, where

$$x_1 = (2,2,1)^\top \quad x_2 = (3,3,2,3,3,2,1,1,0)^\top$$

and choose a_k so that for $k \geq 3$,

$$\begin{aligned} x_k = & 1_k + (1,0,0)^\top \otimes x_{k-1} \\ & + (0,1,0)^\top \otimes (a_k, a_k, 2, 2, a_k, 2, 1, 1, 1)^\top \otimes 1_{k-3}\,. \end{aligned}$$

Here a_k is chosen maximally such that $U_k x_k \geq 0$ and $m_k \geq x_k$.

All rows of H_k have the same sum, and so $p^\top H_k p$ is minimized by $p = (1/3^k) 1_k$, and the minimum value is $p^\top H_k p = 1_k^\top x_k / 3^k$.

So the theorem is true provided $1^\top x_k = 3^k w_k$.

$1^\top x_k = 3^k w_k$ iff we can take

$$a_k = \begin{cases} 3 - \dfrac{1}{3^{(k-3)/2}}, & \text{when } k \text{ is odd,} \\ 3 - \dfrac{2}{3^{(k-2)/2}}, & \text{when } k \text{ is even.} \end{cases}$$

Note that a_k increases monotonically in k, from 2 towards 3. As $k \to \infty$ we find $a_k \to 3$ and $1_k^\top x_k / 3^k \to \frac{5}{2}$.
Finally, we prove that with these a_k we have always have

1. $m_k \geq x_k$, (implying $M_k \geq H_k$).
2. $U_k x_k \geq 0$, (implying $\bar{H}_k \succeq 0$).

Both are proved by induction. The first is easy and the second is hard. To prove the second we use the recurrence relation for x_k to find recurrences relations for components of the vectors $U_k x_k$, and then show that all components are nonnegative. ■

Conjectures

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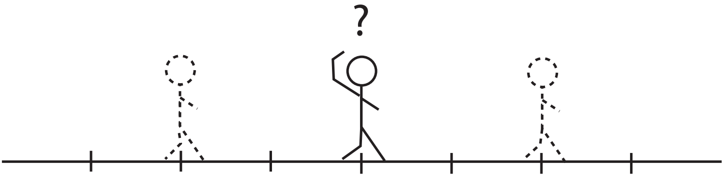
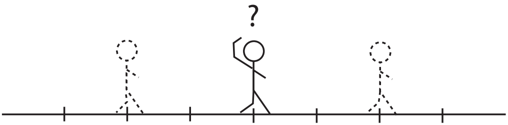
Conjecture: **AW** is asymptotically optimal, in the sense that one can do no better than $ET \sim 0.8289n$.

Symmetric rendezvous search on the line

Two players are placed 2 units apart on a line, randomly facing left or right. At each step each player must either move one unit forward or backwards. Each player knows that the other player is equally likely to be in front or behind him, and equally likely to be facing either way. How can they meet in the least expected time?

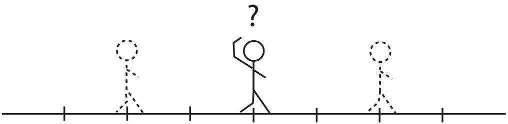


Conjectures



$4.1820 \leq w \leq 4.2574$ (Improve these bounds?)

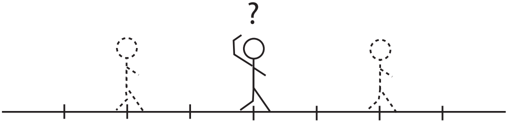
Conjectures



$4.1820 \leq w \leq 4.2574$ (Improve these bounds?)

Conjecture: $w = 4.25$ has been made by Donglei Du.

Conjectures



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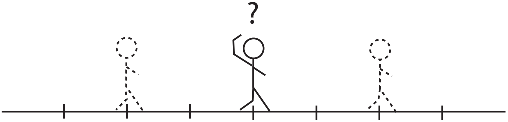
Conjectures



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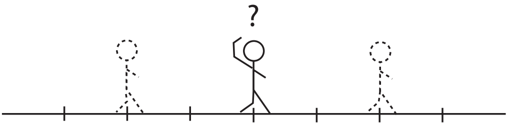
We have seen that on 3 locations it is no help for players to be given a common notion of clockwise. Similarly, here:

Conjectures



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Conjecture: $w = 4.25$ has been made by Donglei Du.

Conjecture: The optimal strategy is not Markovian.

We have seen that on 3 locations it is no help for players to be given a common notion of clockwise. Similarly, here:

Conjecture: it does not to help if players are told that they are initially faced the same way.