Mathematics of Operational Research **Example Sheet 3**

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1. Consider the following bimatrix game:

$(2, 6)$	$(4, 2)$
$(6, 0)$	$(0, 4)$

(a) Compute all equilibria of this game, as well as the maximin strategies for both players. If you were the row player and must go first, what would you do (assuming you know only that the column player will make a best response)?

(b) Discuss your findings in the light of Theorem 15.2.

2. Use the Lemke-Howson algorithm to show that both players receive an expected payoff of 2/3 in the unique equilibrium of the following bimatrix game:

$(0, 1)$	$(1, 0)$	$(0, -1)$
$(2, 0)$	$(0, 2)$	$(-1, 1)$

3. The electorate of a town consists of 51 Labour supporters and 49 Conservative supporters. In an election that is decided by majority vote, the utility to a voter of his party winning is 10, but he suffers a disutility for the inconvenience of going to the polling station of -1. Treating this as a 100-person game, show that neither of the following is an equilibrium for the game: (a) all voters vote; (b) no voter votes.

Show that there is an equilibrium strategy in which no Conservative voter goes to the polling station, and each Labour voter goes to the polling station independently with probability $p = 1 - (1/10)^{1/50} = 0.045007$.

Note that the expected turnout at this equilibrium is less than 2.

4. Prove that if x is the nucleolus of a coalitional game then it has the property that every objection has a counter-objection. An objection can be made by a coalition S claiming that for some y they have less excess, i.e. $e(S, x) > e(S, y)$. There is a counter-objection if some other coalition, T can claim

$$e(T, y) > e(T, x) \quad \text{and} \quad e(T, y) \geq e(S, x).$$

Hint: Assume the nucleolus is unique. Why must there be some T such that $e(T, y) > e(T, x)$? Try taking $T = S_k^x$, where $S_1^x, S_2^x, \ldots$ are the coalitions listed in decreasing order of excess under x and k is the least index such that $e(S_k^x, y) > e(S_k^x, x)$. Suppose $S = S_i^x$ and consider seperately the cases $i \geq k$ and $i < k$.

5. For an undirected graph (V, E) and a weight function $w : E \to \mathbb{R}$, consider the coalitional game with set V of players and characteristic function v given by

$$v(S) = \sum_{\{i,j\} \subseteq S} w(\{i, j\}).$$

Show that the Shapley value of player $i \in V$ in this game is

$$\frac{1}{2} \sum_{j \in V \setminus \{i\}} w(\{i, j\}).$$

6. Consider a jury system with twelve jurors in which a defendant is found guilty if voted guilty by ten or more of the jurors.

(a) Represent this jury system as a coalitional game where $v(S) = 1$ if the defendant is found guilty if voted guilty by all members of S, and $v(S) = 0$ otherwise. Show that the core of this game is empty, and that the nucleolus and the vector of Shapley values are both $(1/12, 1/12, \ldots, 1/12)$.

Now assume that there is a judge in addition to the jurors, and that to be found guilty the defendant in particular has to be voted guilty by the judge.

(b) Represent the new situation as a coalition game, and determine the core, the nucleolus, and the Shapley value of each player.

7. Consider the following bimatrix game:

$(4,1)$	$(1,0)$
$(-1,2)$	$(2,3)$

(a) Determine the Nash bargaining solution of this game when the disagreement point is determined by the two security levels.

(b) Show that the game can be interpreted as a two-player coalitional game with characteristic function v given by $v(\{1\}) = 3/2$, $v(\{2\}) = 1$, and $v(\{1,2\}) = 5$. Determine the core, the nucleolus, and the Shapley value of each player.

8. Show that if in a coalitional game each player receives a payoff equal to his Shapley value then it is true to say: 'The payoff I lose if you leave the game is equal to the payoff you lose if I leave the game.'

Suppose agent i knows about a set of books B_i. If a set of agents S pool what they know then their payoff is the number of books about which they collectively know, i.e. $\left|\sum_{i \in S} B_i\right|$. Show that the game is superadditive and the core is nonempty only if the sets $B_1, \ldots, B_n$ are disjoint. Show that agent i has Shapley value $x_i = \sum_{b \in B_i} |\{k : b \in B_k\}|^{-1}$.

9. Kemeny's rule is a social choice function f, determining $f = \succ$ so that the number of agreements with $\{\succ_i\}_{i \in N}$ is maximized . Consider $n = 7$, $m = 3$. The number of agreements with the possible social preferences is shown. Kemeny's rule selects $\succ b \succ c$.

	3	2	2
	a	b	c
	b	c	a
	c	a	b
$a \succ b \succ c$	9	2	2
$a \succ c \succ b$	6	0	4
$b \succ a \succ c$	6	4	0
$b \succ c \succ a$	3	6	2
$c \succ a \succ b$	3	2	6
$c \succ b \succ a$	0	4	4

Suppose voters of the second type change their preferences between b and c. Show that Kemeny's rule fails independence of irrelevant alternatives.

10. In the formulation of Gibbart-Satterthwaite theorem given in lectures there was an assumption that f is unanimous. Show that a strategyproof SCF is unanimous if an only if it is onto, i.e. that for each alternative $a \in A$ there exists some profile $\{\succ_i\}_{i\in N}$ for which f will select a.

11. Consider a SIPV sealed-bid auction in which the item is awarded to the highest bidder, but all bidders pay their bids (even when unsuccessful). The revenue equivalence theorem states that the expected payment of a bidder who has valuation v and wins with probability $p(v)$ is

$$e(p(v)) = vp(v) - \int_0^v p(w)\,dw.$$

Suppose there are n bidders and their private values are i.i.d. uniformly on $[0, 1]$. Show that the a bidder with private value v has, in equilibrium, the bid $\frac{n-1}{n}v^n$. Explain what is meant by 'in equilibrium'.

Suppose $n = 2$. Find the variance of the sale price. Show that a risk adverse seller will prefer the sealed-bid all-pay auction to a English ascending price auction. Is the same true for $n = 3$?

12. Consider a situation with n agents and k identical copies of an item, and assume that agent i has value $v_i \geq 0$ if it receives at least one copy, and value 0 otherwise. Derive the VCG mechanism with Clarke pivot rule for this problem. Give a succinct description of the mechanism and explain from first principles why it is strategyproof.

13. Residents of an apartment complex have to decide whether to build a shared swimming pool. Resident i has value $v_i \geq 0$ if the pool is built, and value 0 otherwise. The cost of the pool is C, so it should be built if $\sum_i v_i \geq C$.

(a) Consider the VCG mechanism with Clarke pivot rule for this problem. Show that an agent i is charged

$$p_i = \begin{cases} C - \sum_{j\neq i} v_i & \text{if } \sum_i v_i \geq C \text{ and } \sum_{j\neq i} v_j < C \\ 0 & \text{otherwise.} \end{cases}$$

(b) Show that this mechanism does not collect enough money to pay for the pool unless $\sum_i v_i = C$, or unless there is a single resident i such that $v_i \geq C$ and $v_j = 0$, $j \neq i$.

(c) Show that the mechanism is susceptible to collusion, i.e., manipulable by a coalition of residents who change their revealed values such that the pool is still build, but every member of the coalition pays less.

(d) Now assume that in addition to the decision whether the pool is built, we can also decide which residents are allowed to use it. Consider a mechanism that works as follows. Initially, all residents submit their value for the pool. Then the mechanism proceeds in rounds and successively excludes residents from using the pool if it was built. Assume that in a certain round, the set of remaining residents is S. If for all $i \in S$, $v_i \geq C/|S|$, then the pool is built, and each resident in S pays $C/|S|$ and is allowed to use the pool. Otherwise all residents i for which $v_i < C/|S|$ are excluded, and the mechanism continues. If at some point all residents have been excluded, the pool is not built. Show that this mechanism is strategyproof.