Mathematics for OR - Examples 3

1. Solve the two-person zero-sum games having payoff matrices (for the row player) of

$$\text{(i) } A = \begin{pmatrix} 3 & 1 \\ 2 & 0 \end{pmatrix} \qquad \text{(ii) } A = \begin{pmatrix} 4 & 0 \\ 2 & 6 \end{pmatrix}.$$

Also find equilibria of the two-person non-sero-sum games in which $B = A^\top$.

You can check your answers using `http://banach.lse.ac.uk/form.html`. This provides a solver for both zero-sum and non-zero-sum two person games.

2. Consider the two-person bimatrix game defined by the two strictly positive $n \times m$ matrices $A = (a_{ij})$ and $B = (b_{ij})$ in which players I, II have n and m pure strategies. If they play pure strategies i and j respectively, then the payoffs to I and II are a_{ij}, and b_{ij}, respectively. Consider a second bimatrix game defined by two matrices $\bar{A}$ and $\bar{B}$, in which both players have $n + m$ strategies and

$$\bar{A} = \bar{B}^\top = \begin{pmatrix} 0 & A \\ B^\top & 0 \end{pmatrix}.$$

Let $S = \{s \in \mathbb{R}^{n+m},\ s \geq 0,\ \sum_i s_i = 1\}$.

Suppose that $\bar{s} \in S$ is a symmetric equilibrium of this game, so that $s^\top \bar{A}\bar{s} \leq \bar{s}^\top \bar{A}\bar{s}$ for all $s \in S$. Let $\alpha = \sum_{i=1}^{n} \bar{s}_i$ and $\beta = \sum_{i=n+1}^{n+m} \bar{s}_i$. Show that (i) both α and β are positive, and (ii) $(\bar{p}, \bar{q})$ is an equilibrium pair in the original game, where these are $\bar{p} = (1/\alpha)(\bar{s}_1, \dots, \bar{s}_n)$ and $\bar{q} = (1/\beta)(\bar{s}_{n+1}, \dots, \bar{s}_{n+m})$.

3. Consider two bimatrix game,s (i) an (ii), in each of which each of the two players has 3 pure strategies and

$$\text{(i) } A = B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{(ii) } A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad B^\top = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

In game (i) find 7 Nash equilibria.

In game (ii) use the Lemke-Howsen algorithm to find a Nash equilibrium. To get you started, note that the LH algorithm works by moving amongst adjacent solutions to $B^\top x + z = 1$ and $Ay + w = 1$, such that $x_i = w_i = 0$ or $y_i = z_i = 0$ for at most one i and where $(x, w, y, z) = (0, 0, 0, 1, 1, 1, 0, 0, 0, 1, 1, 1)$ initially. Begin by increasing x_1 from 0. This reduces z_2 to 0, and so your second step is to increase y_2. Doing this reduces w_2 to 0, so your third step will be to increase x_2. Continue until $y^\top z = x^\top w = 0$ and neither x or y equals $(0, 0, 0)$.

4. 'Odd Man Out' is a three-player zero-sum game. Three players simultaneously choose heads or tails. If all three make the same choice no money changes hands.

However, if one player chooses different from the others he must pay each of the others £1. What are the Nash equilibrium?

5. The electorate of a town consists of 51 Labour supporters and 49 Conservative supporters. In an election that is decided by majority vote, the utility to a voter of his party winning is 10, but he suffers a disutility for the inconvenience of going to the polling station of -1. Treating this as a 100-person game, show that neither of the following is an equilibrium for the game: (a) all voters vote; (b) no voter votes.

Show that there is an equilibrium strategy in which no Conservative voter goes to the polling station, and each Labour voter goes to the polling station independently with probability $p = 1 - (1/10)^{1/50} = 0.045007$.

Note that the expected turnout in this equilibrium is less than 2.

6. A two-person non-zero-sum game has payoff matrix

$$\begin{pmatrix} (4, 1) & (1, 0) \\ (-1, 2) & (2, 3) \end{pmatrix}.$$

Show that the bargaining set of this game is $\{(u, v) : u + v = 5,\ 2 \leq u \leq 4\}$ and that the maximin bargaining solution is

$$\arg \max_{(u,v) \in B} \left(u - \tfrac{3}{2}\right)(v - 1) = \left(\tfrac{11}{4}, \tfrac{9}{4}\right).$$

Consider this same game a as 2-person coalitional game and show its characteristic function is $v(1) = \frac{3}{2}$, $v(2) = 1$ and $v(1, 2) = 5$. Hence find all the imputations. Show that the Shapley value is equal to the maximin bargaining solution. What is the core and nucleolus for this game?

7. Consider a jury system in which at least ten of the twelve jurors must vote a man guilty for him to be found guilty. Describe the characteristic function of this game where $v(S) = 1$, if, when the members of S vote him guilty, he is sure to be found guilty, and $v(S) = 0$ otherwise. Show that the core of this game is empty.

Show that the Shapley value and the nucleolus are both $(1/12, 1/12, \dots, /12)$.

If the judge believes a defendant is innocent he will direct the jury to find him innocent; only if he believes the defendent guilty will he let the jury decide for themselves. In this 13-person game, what is the characteristic function? Show that the core is $(1, 0, 0, 0, \dots, 0)$ where player 1 is the judge, and that the Shapley value is $(18/78, 5/78, ..., 5/78)$. What is the nucleolus?

8. Show that if in a coalitional game each player receives a payoff equal to his Shapley value then it is true to say: 'The payoff I lose if you leave the game is equal to the payoff you lose if I leave the game.'

Suppose agent i knows about a set of books B_i. If a set of agents S pool what they know then their payoff is the number of books about which they collectively know, i.e. $\left|\sum_{i\in S} B_i\right|$. Show that the game is superadditive and the core is nonempty only if the sets $B_1, \dots, B_n$ are disjoint.

Show that agent i has Shapley value $x_i = \sum_{b\in B_i} |\{k : b \in B_k\}|^{-1}$.

9. Consider a duopoly with zero production costs and where the quantities q_1 and q_2, sold by Firms 1 and 2, respectively, are related to prices by

$$q_1 = \max\{0, 10 + 2p_2 - p_1\}$$
$$q_2 = \max\{0, 20 + p_1 - 3p_2\}$$

(i) Find the Cournot equilibrium.

(ii) Suppose Firm 1 (called the Stackkeberg leader) sets its price first and then Firm 2 (the follower) sets its price. Compare the payoffs obtained by the firms with those they obtained in (i).

Hint: the answers you should get for the revenues are (i) $(100, 75)$ and (ii) $(104.1667, 88.0208)$.

10. Suppose that at the equilibrium of a SIPV auction in which only the winner pays his bid it is optimal for a bidder with private valuation v to bid x_1 when $v = v_1$, and x_2 when $v = v_2$. Suppose that a bid of x_i wins the auction with probability p_i, and $x_1 < x_2$, $p_1 < p_2$. Suppose that the valuations v_1 and v_2 are equally likely, say $P(v = v_i) = \phi_i$, where $\phi_1 = \phi_2$. By considering an alternative strategy in which the bidder bids x_2 when his private value is v_1, and x_1 when it is v_2, show that the first-mentioned bidding strategy can only be optimal if $v_1 \le v_2$.

Note that the above provides an alternative proof of Lemma 22.1 if all valuations are equally likely. Can you extend this method of proof so that is also works when $\phi_1 \neq \phi_2$?

11. Consider a SIPV sealed-bid auction in which the item is awarded to the highest bidder, but all bidders must pay their bids (even when unsuccessful). The revenue equivalence theorem states that the expected amount paid by a bidder who has private value v and wins with probability $p(v)$ is

$$e(p(v)) = vp(v) - \int_0^v p(w)\,dw.$$

Suppose there are n bidders and their private values are i.i.d. uniformly on $[0, 1]$. Show that the a bidder with private value v has, in equilibrium, the bid $\frac{n-1}{n}v^n$. Explain what is meant by 'in equilibrium'.

Suppose $n = 2$. Find the variance of the sale price. Show that a risk adverse seller will prefer the above sealed-bid all-pay auction to a English ascending price auction.

12. Consider an multi-unit auction in which k identical items are to be auctioned to n bidders ($k < n$). Each bidder wishes to obtain one of the items and has an independent private value for so doing, which can be modelled as a random variable with distribution function F. The items are be allocated to the k highest bidders. Prove that the revenue equivalence theorem holds.

Find the seller's expected revenue when $k = 2$, $n = 4$ and F is the uniform distribution on $[0, 1]$.

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