Mathematics for OR - Examples 1

1. Consider the following constrained optimization problem.

$$\begin{array}{ll} \text{maximize} & -2x_1^2 - x_2^2 + x_1x_2 + 8x_1 + 3x_2 \\ \text{subject to} & 3x_1 + x_2 = 10\,. \end{array}$$

Show that the optimal solution is at $(x_1, x_2) = (69/28, 73/28)$.

2. Use the two-phase simplex method to solve the problem

$$\begin{array}{lrl} \text{minimize} & 4x_1 + 4x_2 + x_3 & \\ \text{subject to} & x_1 + x_2 + x_3 & \leq 2 \\ & 2x_1 + x_2 & \leq 3 \\ & 2x_1 + x_2 + 3x_3 & \geq 3 \\ & x_1, x_2, x_3 & \geq 0\,. \end{array}$$

You should find that the optimum occurs at $x = (0, 0, 1)$.

3. Consider the simplex algorithm applied to a linear programming problem $\{\text{maximize } c^\top x : Ax = b,\ x \geq 0\}$. Suppose the rows of A are linearly independent.

For each of these statements give a proof or counterexample.

(a) A variable that has just left the basis cannot reenter at the very next step.

(b) A variable that has just entered the basis cannot leave at the very next step.

4. Show that in Phase I of the two-phase simplex method, if an artificial variable becomes nonbasic it need never become basic again. So as soon as an artificial variable is nonbasic its column can be eliminated from the tableau.

5. Consider the pair of linear programs: $P = \text{maximize}\,\{0^\top x :\ Ax = b,\ x \geq 0\}$ and $D = \text{minimize}\,\{y^\top b :\ y^\top A \geq 0^\top\}$. Show that D is the dual of P.

Show that P is feasible if and only if D is bounded.

Prove Farkas' lemma, which states that exactly one of the following must hold:

(a) There exists some vector $x \geq 0$ such that $Ax = b$.

(b) There exists some y such that $y^\top A \geq 0$, $y^\top b < 0$.

6. Consider the linear program

$$P: \quad \text{maximize } c^\top x\,, \quad Ax \leq b \text{ and } x \geq 0.$$

Here $c, x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ and A is $m \times n$. Derive the dual problem, D.

Prove or provide a counterexample to each of the following.

(a) If P is unbounded then D is infeasible.

(b) If P is infeasible then D is unbounded.

The following is the final simplex algorithm tableau for a linear programming problem P, in which $n = m = 2$. Find the all optimal solutions to both the primal and the dual problems.

What was the original primal problem?

x_1	x_2	z_1	z_2	
0	1	$\frac{1}{10}$	$\frac{1}{10}$	1
1	0	$\frac{1}{20}$	$\frac{3}{10}$	2
0	0	0	$-\frac{1}{2}$	-3

7. Consider the following integer linear programming problem (ILP)

$$\begin{array}{lrcrl} \text{maximize} & x_1 & + & 2x_2 & \\ \text{subject to} & -3x_1 & + & 4x_2 & \leq 4 \\ & 3x_1 & + & 2x_2 & \leq 11 \\ & 2x_1 & - & x_2 & \leq 5 \end{array}$$

$x_1, x_2 \geq 0,\ x_1, x_2$ integer.

Use the simplex method to solve the LP relaxation (i.e. when x_1, x_2 need not be integers), verifying that the final tableau is

x_1	x_2	z_1	z_2	z_3	
0	1	1/6	1/6	0	5/2
1	0	$-1/9$	2/9	0	2
0	0	7/18	$-5/18$	1	7/2
0	0	$-2/9$	$-5/9$	0	-7

Argue that in the optimal solution to the ILP we must have $x_2 \leq 2$.

Use Gomory's method (with the dual simplex algorithm) to solve the ILP.

8. Show that any instance of the **Satisfiability** decision problem can be reduced to an instance of 0–1 linear programming (i.e. a linear program in which every decision variable must be 0 or 1). Show how this reduction could be made for the satifiability instance: *Is there an assignment of the values 'true' and 'false' to the boolean variables $X_1, X_2, X_3, X_4, X_5, X_6$ which makes the following true?*

$$(X_1 \vee \bar{X}_2 \vee X_6) \wedge (\bar{X}_2 \vee \bar{X}_4) \wedge (X_3 \vee X_5 \vee X_6)\,,$$

where $\wedge$ means AND, $\vee$ means OR, and $\bar{X}_i$ means 'not X_i'.

Given that **Satisfiability** is $\mathcal{NP}$-complete, show that 0-1 integer programming is also $\mathcal{NP}$-complete.

9. Let G be a graph with n nodes. Consider two decision problems.

Clique: *does there exist a clique of size k in G?* (i.e. k vertices such that there is an edge between every pair).

Vertex Cover: *is there a vertex cover of size k?* (i.e. as set of k nodes such that every edge in G has an endpoint in this subset).

Let G' be the complement of G, i.e. G' has an edge between two nodes if and only if G does not. Show G has a clique of size k if and only if G' has a vertex cover of size $n-k$.

Given that **Clique** is $\mathcal{NP}$-complete, prove that **Vertex Cover** is $\mathcal{NP}$-complete.

10. **2-SAT** is the decision problem of whether or not it is possible to assign values of true (T) and false (F) to k variables $x_1, \dots, x_k$ so that a given conjuction of m clauses is true. I.e. so that a given

$$(x_{11} \vee x_{12}) \wedge (x_{21} \vee x_{22}) \wedge \cdots \wedge (x_{m1} \vee x_{m2}) \qquad (1)$$

is true, where $\wedge$ means AND, $\vee$ means OR, each x_{ij} is one of $x_1, \dots, x_k$, with or without a NOT in front of it, and each variable can appear multiple times in the expression. Show that **2-SAT** $\in \mathcal{NP}$.

Let $\bar{x}$ denote 'NOT x'. Suppose we create an *implication graph* with vertex set $V = \{x_1, \dots, x_k\} \cup \{\bar{x}_1, \dots, \bar{x}_k\}$. We put a directed edge (x, y) in the graph iff $(\bar{x}, y)$ is equivalent to one of the clauses $(x_{i1} \vee x_{i2})$. Show the following.

(a) If (z_1, z_2) is an edge, so is $(\bar{z}_2, \bar{z}_1)$.

(b) If $x_1, \dots, x_k$ take values such that (1) is true and there is directed path in the graph $(z_1, z_2, \dots, z_j)$, then $z_1 =$T implies $z_2 = \cdots = z_j =$T.

(c) Consider the statement (2) 'there exists a variable z such that there is a path from z to $\bar{z}$ and a path from $\bar{z}$ to z'.

  (i) Show that if (2) is true then (1) is not satisfiable.

  (ii) Show that if (2) is not true then (1) can be made satisfiable by the following polynomial time algorithm:

  Pick an unassigned variable x, in some clause, for which there is no directed path from x to $\bar{x}$. Assign $x =$T (making that clause true), and also assign T to all vertices that can be reached along paths from x. Assign F to the negations of all variables that are so assigned. Repeat this until all variables have been assigned.

  (You must show that the algorithm never sets both $y =$T and $\bar{y} =$T, and that if (x, y) is clause the algorithm cannot set both $x = F$ and $y = F$.)

(d) Deduce that 2-SAT$\in \mathcal{P}$.

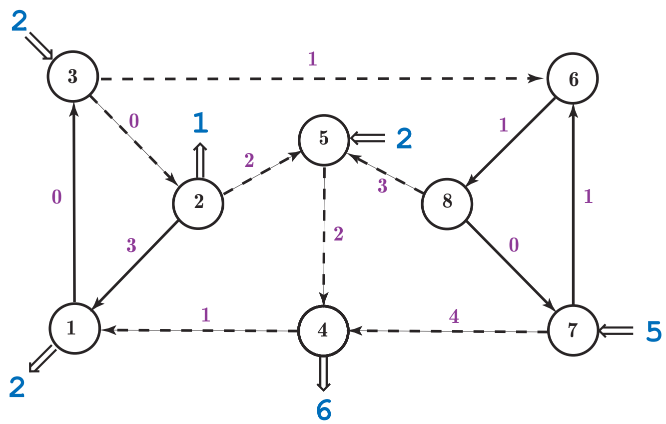
11. **3-SAT** is like **2-SAT** except that the clauses are of the form $(x_{i1} \vee x_{i2} \vee x_{i3})$. Suppose we construct a graph G with vertex set $\{< x, i >: x \text{ is in the } i\text{th clause}\}$. We place an undirected edge $\{< z_1, i >, < z_2, j >\}$ iff $z_1 \neq \bar{z}_2$, and $i \neq j$.

Show that

(a) The **3-SAT** instance is satisfiable iff G has a clique of size m.

(b) Consider the **Clique** decision problem (CDP); *Does there exist a clique of size m in graph G?* (i.e. m vertices such that there is an edge between every pair). Show that if **3-SAT** is $\mathcal{NP}$-complete then CDP is $\mathcal{NP}$-complete.

12. Consider the uncapacitated network flow problem below. The label next to each arc is its cost, c_{ij}. Use the network simplex algorithm to find the minimum cost flow. Start with the tree indicated by the dashed lines in the figure.



Richard Weber, 18/10/10