Mathematics for Operations Research Examples 3

1. Solve the two-person zero-sum games with the payoff matrices (for player 1) of

$$(a)\ \begin{pmatrix} 3 & 1 \\ 2 & 0 \end{pmatrix} \qquad (b)\ \begin{pmatrix} 4 & 0 \\ 2 & 6 \end{pmatrix}$$

2. 'Odd Man Out' is a three-player zero-sum game. Three players simultaneously choose heads or tails. If all three make the same choice no money changes hands. However, if one player chooses different from the others he must pay each of the others £1. What are the Nash equilibrium?

3. The electorate of a town consists of 51 Labour supporters and 49 Conservative supporters. In an election that is decided by majority vote, the utility to a voter of his party winning is 10, but he suffers a disutility for the inconvenience of going to the polling station of -1. Treating this as a 100-person game, show that neither of the following is an equilibrium for the game: (a) all voters vote; (b) no voter votes.

   Show that there is an equilibrium strategy in which no Conservative voter goes to the polling station, and each Labour voter goes to the polling station independently with probability $p = 1 - (1/10)^{1/50} = 0.045007$. Note that the expected turnout in this equilibrium is only about 6.

4. A two-person non-zero-sum game has payoff matrix

$$\begin{pmatrix} (4,1) & (1,0) \\ (-1,2) & (2,3) \end{pmatrix}.$$

   Show that the bargaining set of this game is $\{(u,v) : u+v=5,\ 2 \le u \le 4\}$ and that the maximin bargaining solution is $(\frac{11}{4}, \frac{9}{4})$.

   Consider this same game a as 2-person coalitional game and show its characteristic function is $v(1) = \frac{3}{2}$, $v(2) = 1$ and $v(1,2) = 5$. Hence find all the imputations. Show that the Shapley value is equal to the maximin bargaining solution. What is the core and nucleolus for this game?

5. Consider the nucleolus of the oil market game in Lecture 20. Does it make sense to you that Country 2 is awarded no reward and that Country 1 gets more reward than Country 3? Give an explanation as to why when $b = c$, Country 1 should get all the reward.

6. Consider a jury system in which at least ten of the twelve jurors must vote a man guilty for him to be found guilty. Describe the characteristic function of this game where $v(S) = 1$, if, when the members of S vote him guilty, he is sure to be found guilty, and $v(S) = 0$ otherwise. Show that the core of this game is empty.

   Show that the Shapley value and the nucleolus are both $(1/12, 1/12, \ldots, /12)$.

   If the judge believes a defendant is innocent he will direct the jury to find him so, which they have to do; only if he believes him guilty will he let the jury decide for themselves. In this 13-person game, what is the characteristic function? Show that the core is $(1, 0, 0, 0, \ldots, 0)$ where player 1 is the judge, and that the Shapley value is $(18/78, 5/78, ..., 5/78)$. What is the nucleolus?

7. Consider a duopoly with zero production costs and where the quantities q_1 and q_2 produced by the two firms are related to prices by

$$q_1 = \max\{0, 10 + 2p_2 - p_1\}$$
$$q_2 = \max\{0, 20 + p_1 - 3p_2\}$$

   Find the following

   (i) the Cournot equilibrium solution;

   (ii) the two Stackleberg solutions;

   Compare these solutions by clearly stating the correspondiong prices p_1, p_2 and the quantities q_1, q_2 and the resulting revenues of the firms.

   Hint: the answers you should get for the revenues are (i) $(100, 75)$ and (ii) $(104.1667, 88.0208)$.

8. Some games have no ESS, and some games have more than one ESS. Show these facts by looking for ESS of the games

$$(a)\ \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \qquad (b)\ \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

9. Consider a SIPV auction. Suppose that in equilibrium it is optimal for a bidder to bid x_1 when his private value is v_1, and x_2 when it is v_2. Suppose that valuations v_1 and v_2 are equally likely. By comparing this to an alternative (and supposedly nonoptimal) strategy, in which he bids x_2 when his private value is v_1, and x_1 when it is v_2, show that we must have $x_1 < x_2$ if $v_1 < v_2$. Note that this gives an alternative proof of Lemma 23.1.

10. Consider a SIPV sealed-bid auction in which the item is awarded to the highest bidder, but all bidders must pay their bids (even when unsuccessful). The Revenue Equivalence Theorem states that the expected amount paid by a bidder who has private value v and wins with probability $p(v)$ is

$$e(p(v)) = vp(v) - \int_0^v p(w)\,dw\,,$$

Suppose there are n bidders and their private values are i.i.d. uniformly on $[0,1]$. Show that the a bidder with private value v has, in equilibrium, the bid $\frac{n-1}{n}v^n$. Explain what is meant by 'in equilibrium'.

Suppose $n=2$. Find the variance of the sale price. Will a risk adverse seller prefer this auction or an English auction?

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