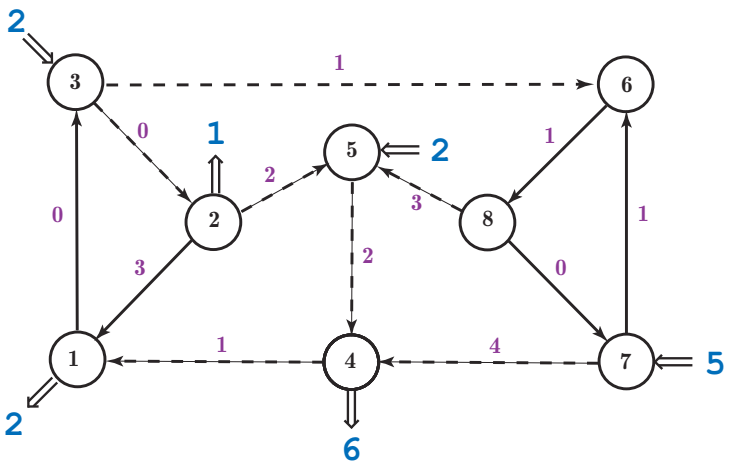
Mathematics for Operations Research - Examples 2

1. Consider the uncapacitated network flow problem below. The label next to each arc is its cost, c_{ij}. Use the network simplex algorithm to find the minimum cost flow. Start with the tree indicated by the dashed lines in the figure.



2. Consider a minimum cost network flow problem in which we impose the additional constraint $f_{ij} \geq d_{ij}$ for every arc (i, j). Construct an equivalent network flow problem in which there are no nonzero lower bounds on the arc flows. Hint: Let $\bar{f}_{ij} = f_{ij} - d_{ij}$ and construct a new network for the arc flows $\bar{f}_{ij}$. How should b_i be changed?

3. Consider a transportation problem in which all shipping costs c_{ij} are positive. Suppose we increase the supply at some sources and increase the demands at some sinks. To maintain feasibility we ensure that the total supply remains equal to the total demand. Is it true that the value of the optimal cost will increase? Prove this, or provide a counterexample.

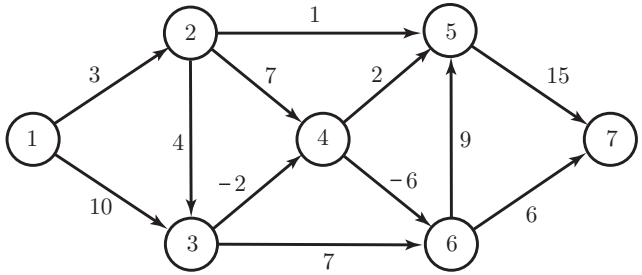
4. Each of n teams plays against each other team k games (so $kn(n-1)/2$ games are played in all). Assume that every game ends in a win or a loss (no draws) and let x_i be the total number of wins of team i. Let X be the set of all possible outcome vectors $(x_1, \ldots, x_n)$. Given an arbitrary vector $(x_1, \ldots, x_n)$ we want to determine whether it belongs to X, that is, whether it is a possible tournament outcome vector. Provide a network formulation of this problem.

5. Consider the following algorithm, applied to a graph with n nodes in which the distance between nodes i and j is c_{ij}.

   0. Set $k = 1$. Arbitrarily choose a node, say 1, and let $N_1 = \{1\}$.
   1. For each $j \notin N_k$, find the node $i \in N_k$ that minimizes c_{ij}. Call this node $v_k(j)$.
   2. Let j_k^* be the node that minimizes $c_{j,v_k(j)}$ over $j \notin N_k$. Set $N_{k+1} = N_k \cup \{j_k^*\}$.
   3. Stop if $k + 1 = n$; otherwise set $k = k + 1$ and goto step 1.

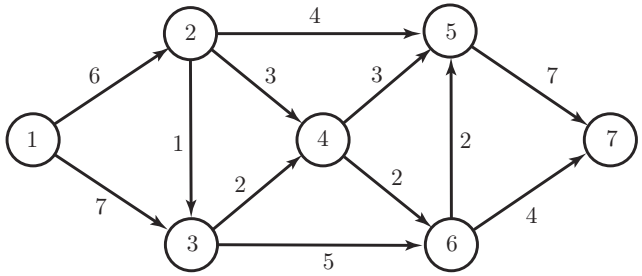
   What problem does this algorithm solve? Show that its running time is $O(n^2)$.

6. Use the Bellman-Ford algorithm to find the least cost path from every node to node 7 in the following network. Note that some arcs have negative cost.



   Now use Dijkstra's algorithm on a modified network to solve the all-pairs problem.

7. Find the maximal flow from node 1 to node 7 in the following network. The numbers by the arcs are their capacities.



8. A hiker wishes to choose items to take on a journey such that the total value of the items is at least 9, but the total weight is a minimum. Solve this problem using a branch and bound approach.

i	1	2	3	4	5
v_i	5	5	4	2	3
w_i	5	6	3	1	2

9. Let $M = \{1, 2, \ldots, n\}$ be a set of n items and let $M_1, M_2, \ldots, M_m$ be a collection of m subsets of M. let F be a subset of $\{1, \ldots, m\}$. We say that a F is a **cover** if $\cup_{i \in F} M_i = M$ and say that F is a **packing** if $M_i \cap M_j = \emptyset$ for all $i, j \in F$. We are given a weight c_i for each M_i, and the weight of F is $\sum_{i \in F} c_i$. Let $A_{ij} = 1$ or 0 as M_i does or does not contain item j. Let x_i be 1 or 0 as i is or is not in F. Express as an integer linear programs the problems of finding: (a) the cover of minimum weight, and (b) the packing of maximum weight.

   Show that the LP relaxation of (a) has the dual problem:

$$\text{(c): maximize } \sum_{j=1}^{n} y_j\,, \quad \text{subject to } \sum_{j=1}^{n} A_{ij} y_j \le c_i, \quad \text{for all } i, \text{ and } y \ge 0.$$

   Suppose all c_j are integers. Consider the problem

$$\text{(d): maximize } |S|\,, \quad \text{subject to } S \subseteq M \text{ and } |S \cap M_i| \le c_i \text{ for all } i\,,$$

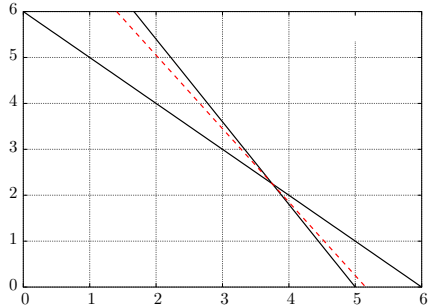
   where $|S|$ denotes the number of items in S. Show that the solution value to (d) is never more than the solution value to (a). Can the inequality be strict?

   Explain how you could use (d) in a branch and bound algorithm for solving (a).

10. Use Dakin's method to solve the following ILP.

$$\begin{aligned} \text{maximize } z = {} & 8x_1 + 5x_2 \\ \text{subject to } & x_1 + x_2 \le 6 \\ & 9x_1 + 5x_2 \le 45 \\ & x_1, x_2 \ge 0 \\ & x_2, x_2 \text{ integer} \end{aligned}$$

    You need not carry out iterations of the Simplex method. Rather, explain how Dakin's method will proceed by referring to the following plot.



11. Given these distances between 5 towns, find a minimum spanning tree.

	A	B	C	D	E
A		9	7	5	7
B	9		9	9	8
C	7	9		7	6
D	5	9	7		6
E	7	8	6	6	

    By removing town B and finding a minimum spanning tree for the remaining towns, show that a lower bound on the solution to the travelling salesman problem for this graph is $9 + 8 + 17 = 34$.

    Compare this to what you obtain by starting at town A and using a greedy algorithm (i.e., always visiting the closest town not yet visited).

12. Suppose A is a heuristic for the TSP. Let $A(I)$ be the length of the tour produced by A on an instance I and let $\mathrm{OPT}(I)$ be the length of the optimal tour. Suppose A is a ϵ-approximation algorithm, running in polynomial time, so that $\mathrm{A}(I) \le (1 + \epsilon)\,\mathrm{OPT}(I)$, for all I.

    Suppose we have a graph $G = (N, A)$, with n nodes, and want to decide if it contains a Hamiltonian cycle (i.e., a tour that visits every node exactly once). Let us construct an instance of TSP in which $c_{ij} = 1$ or $c_{ij} = (1 + \epsilon)n$ as G does or does not contain the edge (i, j). Show that, applied to this instance, $\mathrm{A}(I) \le (1 + \epsilon)n$ if G has a Hamiltonian cycle, and $\mathrm{A}(I) \ge (n - 1) + (1 + \epsilon)n$ if G does not. Hence show that A can be used to decide the Hamiltonian cycle decision problem in polynomial time.

    Given that the Hamiltonian cycle decision problem is $\mathcal{NP}$-complete, what do you conclude about the plausibility that there exists an algorithm like A? Why does this not contradict our derivation of a 1-approximation algorithm in Lecture 16?

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