# Mathematics for Operations Research
# Examples 1

1. Consider the following constrained optimization problem.

$$\begin{array}{ll} \text{maximize} & -2x_1^2 - x_2^2 + x_1x_2 + 8x_1 + 3x_2 \\ \text{subject to} & 3x_1 + x_2 = 10\,. \end{array}$$

   Show that the optimal solution is at $(x_1, x_2) = (69/28, 73/28)$.

2. Use the two-phase simplex method to solve the problem

$$\begin{array}{lrl} \text{minimize} & 4x_1 + 4x_2 + x_3 & \\ \text{subject to} & x_1 + x_2 + x_3 & \le 2 \\ & 2x_1 + x_2 & \le 3 \\ & 2x_1 + x_2 + 3x_3 & \ge 3 \\ & x_1, x_2, x_3 & \ge 0\,. \end{array}$$

   You should find that the optimum occurs at $x = (0, 0, 1)$.

3. Consider the simplex algorithm applied to a linear programming problem with feasible set $\{x : Ax = b,\ x \ge 0\}$. Suppose the rows of $A$ are linearly independent. For each of these statements give a proof or counterexample.

   (a) A variable that has just left the basis cannot reenter at the very next step.

   (b) A variable that has just entered the basis cannot leave at the very next step.

4. Show that in Phase I of the two-phase simplex method, if an artificial variable becomes nonbasic it need never become basic again. So as soon as an artificial variable is nonbasic its column can be eliminated from the tableau.

5. Consider the pair of linear programs: $P = \text{maximize}\,\{0^\top x : Ax = b,\, x \ge 0\}$ and $D = \text{minimize}\,\{y^\top b : y^\top A \ge 0^\top\}$. Show from first principles that $D$ is the dual of $P$.

   Show that $P$ is feasible if and only if $D$ is bounded.

   Prove Farkas' lemma, that exactly one of the following must hold:

   (a) There exists some vector $x \ge 0$ such that $Ax = b$.

   (b) There exists some $y$ such that $y^\top A \ge 0$, $y^\top b < 0$.



6. Consider the following integer linear programming problem (ILP)

$$\begin{array}{lrcrl} \text{maximize} & x_1 & + & 2x_2 & \\ \text{subject to} & -3x_1 & + & 4x_2 & \le 4 \\ & 3x_1 & + & 2x_2 & \le 11 \\ & 2x_1 & - & x_2 & \le 5 \\ \multicolumn{5}{l}{x_1, x_2 \ge 0,\ x_1, x_2 \text{ integer.}} \end{array}$$

   Use the simplex method to solve the LP relaxation (i.e., when $x_1, x_2$ need not be integers), showing that the final tableau is

| $x_1$ | $x_2$ | $z_1$ | $z_2$ | $z_3$ | |
|---|---|---|---|---|---|
| 0 | 1 | 1/6 | 1/6 | 0 | 5/2 |
| 1 | 0 | −1/9 | 2/9 | 0 | 2 |
| 0 | 0 | 7/18 | −5/18 | 1 | 7/2 |
| 0 | 0 | −2/9 | −5/9 | 0 | −7 |

   Argue that in the optimal solution to the ILP we must have $x_2 \le 2$.

   Use Gomory's method (with the dual simplex algorithm) to solve the ILP.

7. Show that any instance of the **Satisfiability** decision problem can be reduced to an instance of 0–1 linear programming (i.e., a linear program in which every decision variable must be 0 or 1). Show how this reduction could be made for the satifiability instance: *Is there an assignment of the values 'true' and 'false' to the boolean variables $X_1, X_2, X_3, X_4, X_5, X_6$ which makes the following true?*

$$(X_1 \vee \bar{X}_2 \vee X_6) \wedge (\bar{X}_2 \vee \bar{X}_4) \wedge (X_3 \vee X_5 \vee X_6)\,,$$

   where $\wedge$ means AND, $\vee$ means OR, and $\bar{X}_i$ means 'not $X_i$'. Given that **Satisfiability** is $\mathcal{NP}$-complete, show that 0-1 integer programming is also $\mathcal{NP}$-complete.

8. Let $G$ be a graph with $n$ nodes. Consider two problems. **Clique**: *is there a subset of $k$ nodes with an edge between every two nodes in the subset?* **Vertex Cover**: *is there a vertex cover of size $k$ (i.e., as set of $k$ nodes such that every edge in $G$ has an endpoint in this subset)?* Let $G'$ be the complement of $G$, i.e., $G'$ has an edge between two nodes if and only if $G$ does not. Show $G$ has a clique of size $k$ if and only if $G'$ has a vertex cover of size $n - k$. Given that **Clique** is $\mathcal{NP}$-complete, prove that **Vertex Cover** is $\mathcal{NP}$-complete.

9. **2-SAT** is the decision problem of whether or not it is possible to assign values of true (T) and false (F) to a set of variables $x_1, \ldots, x_k$ so the following is true

$$(x_{11} \vee x_{12}) \wedge (x_{21} \vee x_{22}) \wedge \cdots \wedge (x_{m1} \vee x_{m2}) \qquad (1)$$

where $\wedge$ means AND, $\vee$ means OR, each $x_{ij}$ is one of $x_1, \dots, x_k$, with or without a NOT in front of it, and each variable can appear multiple times in the expression.

Let $\bar{x}$ denote 'NOT $x$'. Suppose we create a graph with vertex set $V = \{x_1, \dots, x_k\} \cup \{\bar{x}_1, \dots, \bar{x}_k\}$. We put a directed edge $(x, y)$ in the graph iff $(\bar{x}, y)$ is equivalent to one of the clauses $(x_{i1} \vee x_{i2})$. Show the following.

(a) If $(z_1, z_2)$ is an edge, so is $(\bar{z}_2, \bar{z}_1)$.

(b) If $x_1, \dots, x_k$ take values such that (1) is true then $z_1$ =T must imply $z_2 = \cdots = z_j$ =T, if there is path in the graph $(z_1, z_2, \dots, z_j)$.

(c) Consider the statement (2): 'there exists a variable $z$ such that there is a path from $z$ to $\bar{z}$ and a path from $\bar{z}$ to $z$'.

  (i) Show that if (2) is true then (1) is not satisfiable.

  (ii) Show that if (2) is not true then (1) can be made satisfiable by the following polynomial time algorithm:
  Pick an unassigned variable $x$ in some clause. Assign $x$ =T (making that clause true), and also assign T to all vertices that can be reached along paths from $x$. Assign F to the negations of all variables that are so assigned. Repeat this until all variables have been assigned. (You must show that the algorithm never sets both $y$ =T and $\bar{y}$ =T.)

(d) Deduce that 2-SAT$\in \mathcal{P}$.

10. **3-SAT** is like **2-SAT** in the previous question, except that the $m$ clauses are of the form $(x_{i1} \vee x_{i2} \vee x_{i3})$. Suppose we construct a graph $G$ with vertex set $\{< x, i > : \ x \text{ is in the } i\text{th clause}\}$. We place an undirected edge $\{< z_1, i >, < z_2, j >\}$ iff $z_1 \neq \bar{z}_2$, and $i \neq j$.

Show that

(a) The **3-SAT** instance is satisfiable iff $G$ has a clique of size $m$.

(b) Consider the **Clique** decision problem (CDP); *Does there exist a clique of size $m$ in graph $G$?* (i.e., $m$ vertices such that there is an edge between every pair). Show that if **3-SAT** is $\mathcal{NP}$-complete then CDP is $\mathcal{NP}$-complete.

11. The remaining (optional) questions are for students interested in thinking about the Hirsch conjecture. Its relevance is to the question of how fast the simplex algorithm could move from an initial basic feasible solution to an optimal one if pivots are luckily chosen so that this happens as fast as possible.

The **Hirsch conjecture** (1957) concerns polytopes, i.e., convex bodies defined as $P = \{x \ : Ax \leq b\}$. Let $\Delta(d, n)$ be the maximum diameter of any polytope



with $n$ facets (sides) in dimension $d$. The conjecture is that $\Delta(d, n) \leq n - d$. It is known to be false for unbounded polytopes. So we restrict attention to bounded ones. The Hirsch conjecture is known to be equivalent to the ***d*-step conjecture** that $\Delta(d, 2d) = d$. Note that $n = 2d$ is what happens in the cube $\{x \in \mathbb{R}^d \ : 0 \leq x_i \leq 1\,, \ i = 1, \dots, d\}$. It is also equivalent to the conjecture that if all vertices are the intersection of just $d$ faces then any two vertices can be joined by a path of length $\leq n - d$ that does not revisit any face. The Hirsch conjecture has been proved (for bounded polytopes) for $d \leq 6$.

Can you prove the Hirsch conjecture for $d = 2$?

12. In the assignment problem, we are given $m^2$ numbers $\{c_{ij}\}$, and are to

$$\text{maximize} \sum_{i=1}^{m} \sum_{j=1}^{m} c_{ij} x_{ij}\,, \text{ such that}$$

$$\sum_{j=1}^{m} x_{ij} = 1\,, \quad \text{for all } i\,; \quad \sum_{i=1}^{m} x_{ij} = 1\,, \quad \text{for all } j\,; \quad x_{ij} \in \{0, 1\}\,, \quad \text{for all } i, j.$$

Explain why the problem is unchanged if the constraint is changed to $x_{ij} \in [0, 1]$ for all $i, j$. (Hint: Lecture 12).

We claim the following facts. You need not prove them; but may think about why they are true. (i) The extreme points of the feasible set are in 1–1 correspondance with permutations of the set $S = \{1, \dots, m\}$. (ii) Two extreme points are neighbours if and only if the permutations to which they correspond differ by a cyclic permutation of some one subset of $S$. E.g. with $m = 4$, $(1, 2, 3, 4)$ and $(2, 3, 1, 4)$ are neighbours since $(2, 3, 1, 4) = \{(1, 2, 3), (4)\}$, but $(1, 2, 3, 4)$ and $(2, 1, 4, 3)$ are not neighbours since $(2, 1, 4, 3) = \{(1, 2), (3, 4)\}$.

In the assignment problem, $n = m^2 + 2m - 1$ and $d = m^2$, so the Hirsch conjecture would suggest the diameter is no more than $2m - 1$. In fact, the diameter of the assignment problem polytope is only 2. This remarkable fact was proved by Balinski and Russakoff (1974). Prove it yourself (more simply than B&R) by using the following facts. (iii) Every permutation is the product of disjoint cycles: e.g., $p = \{(j_1, \dots, j_{n_1}), (j_{n_1+1}, \dots, j_{n_2}), \cdots, (j_{n_{k-1}+1}, \dots, j_{n_k})\}$. (iv) Every permutation is the product of the two cycles: e.g., $p$ is the product of $(j_{n_k}, j_{n_{k-1}}, \dots, j_{n_1})$ and $(j_1, j_2, \dots, j_{n_k})$.

Suppose $m = 4$. Prove that starting at any extreme point it is possible to reach the optimum extreme point (where $\sum_{ij} c_{ij} x_{ij}$ is maximized) in no more than 2 steps, at each step moving to a neighbouring extreme point at which the value of the objective function is not less (so the so-called 'monotone Hirsch conjecture' is true for this problem). Is the same true for $m = 5$?

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