UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 1 June 2001 1.30 4.30

PAPER 34

MATHEMATICS OF OPERATIONAL RESEARCH

*Attempt **FOUR** questions*

*There are **six** questions in total*

The questions carry equal weight

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Write an essay on bargaining. Your account should include a description of the terms jointly dominated, Pareto optimal, bargaining (or negotiation) set, Nash arbitration procedure and maximin bargaining solution.

2 Consider the optimization problem

$$\min f(x)$$

subject to $h(x) = b,\ x \in X \subset \mathbb{R}^n, b \in \mathbb{R}^m$. Define the Lagrangian function for this problem and then state and prove the Lagrangian Sufficiency Theorem. Define the function ϕ by

$$\phi(b) = \inf_{x \in X : h(x) = b.} f(x)$$

Define the Strong Lagrangian property and show that the following are equivalent

(1) there exists a non-vertical supporting hyperplane to ϕ at b

(2) the problem is Strong Lagrangian.

A company is planning to spend £a on advertising. It costs £3,000 per minute to advertise on television and £1,000 per minute to advertise on radio. If the company buys x minutes of television advertising and y minutes of radio advertising, its revenue in thousands of pounds is given by $f(x, y) = -2x^2 - y^2 + xy + 8x + 3y$. How can the company maximise its revenue? Compare the increase in revenue for each additional advertising pound when a = 1,000 with the case when a = 10,000.

3 Consider the general class of linear programmes given by

$$\min \quad c^T x$$

$$\text{subject to } Ax = b,\ x \geq 0$$

where $x \in \mathbb{R}^n, b \in \mathbb{R}^m$ and where all the entries in A, b and c have absolute magnitudes bounded by $U < \infty$.

Show that such linear programmes can be reduced to the special form

$$\begin{aligned} \min\ & c^{*T} y \\ \text{subject to } & A^* y = 0 \\ & 1^T y = 1 \\ & y \geq 0 \end{aligned}$$

with the additional properties that

(i) $y^{(0)} = (1/n^*, \ldots, 1/n^*)^T$ is feasible (where $y \in \mathbb{R}^{n^*}$)

(ii) the optimal value of the objective is zero.

Why is this a useful result?

4 Consider a network with n nodes and set of arcs A. Let $c_{ij} > 0$ for $(i,j) \in A$ be the length of arc (i,j) and set $c_{ij} = \infty$ if $(i,j) \notin A$. Regarding n as the root node, define the all-to-one shortest path problem. Define the Bellman-Ford algorithm for solving this problem. Discuss why this is referred to as a label-correcting algorithm.

Define v_i to be the shortest path length from node i to node n. Suppose that $j \neq n$ is a node such that $c_{jn} = \min_{i \neq n} c_{in}$. Show that $v_j = c_{jn}$ and $v_j \leq v_k$ for all nodes $k \neq n$. Define Dijkstra's algorithm for the all-to-one shortest path problem. Discuss why this is referred to as a label-setting algorithm. Apply Dijkstra's algorithm to the following network with root node $n = 4$,

where the numbers beside the arcs denote the arc's length.

5 The payoff matrix for a two-person non-zero sum game is

$$\begin{array}{cc} & \begin{array}{cc} II_1 & II_2 \end{array} \\ \begin{array}{c} I_1 \\ I_2 \end{array} & \begin{pmatrix} (3, & 8) & (4, & 4) \\ (2, & 0) & (0, & 6) \end{pmatrix} \end{array}$$

Find all equilibrium pairs when considered as a non-cooperative game. Then find the maximin bargaining solution when the game is considered as a cooperative game. Which game would II prefer to play?

6 Consider the game with characteristic function $v(1) = 1,\ v(2) = 2,\ v(3) = 3,\ v(1,2) = 3,\ v(1,3) = 10,\ v(2,3) = 6$ and $v(1,2,3) = 12$.

Define

(a) the set of imputations

(b) the core

(c) the nucleolus

and find them for the game defined above.

UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 30 May 2002 1.30 to 4.30

PAPER 34

MATHEMATICS OF OPERATIONAL RESEARCH

*Attempt **FOUR** questions*

*There are **six** questions in total*

The questions carry equal weight

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Consider the ILP

$$\begin{aligned} \text{minimize} \quad & 3x_1 + 4x_2 \\ \text{subject to} \quad & \\ & 3x_1 + x_2 \geqslant 4 \\ & x_1 + 2x_2 \geqslant 4 \\ & x_1, x_2 \geqslant 0 \\ & x_1, x_2 \text{ integer} \end{aligned}$$

Ignoring the integer constraints, the following tableau gives the optimal solution.

x_1	x_2	z_1	z_2	
1	0	$-\frac{2}{5}$	$-\frac{1}{5}$	$\frac{4}{5}$
0	1	$\frac{1}{5}$	$-\frac{3}{5}$	$\frac{8}{5}$
0	0	$-\frac{2}{5}$	$-\frac{9}{5}$	$\frac{44}{5}$

Explain Gomory's cutting plane method, illustrating it by showing that from the above tableau one can deduce that the optimal integer solution must satisfy the additional constraint

$$\tfrac{1}{5}z_1 + \tfrac{2}{5}z_2 \geqslant \tfrac{3}{5}.$$

Use this constraint and the dual simplex algorithm to find the optimal solution to the ILP.

2 Let A be a $m \times n$ matrix of integers and let b be a vector in $\mathbb{R}^m$. Let U be the largest absolute value of the entries of A and b. By using Cramer's rule or otherwise, prove that every extreme point of the polyhedron $P = \{x \in \mathbb{R}^n : Ax \geqslant b\}$ satisfies

$$-n!U^n \leqslant x_j \leqslant n!U^n, \quad j = 1, \ldots, n.$$

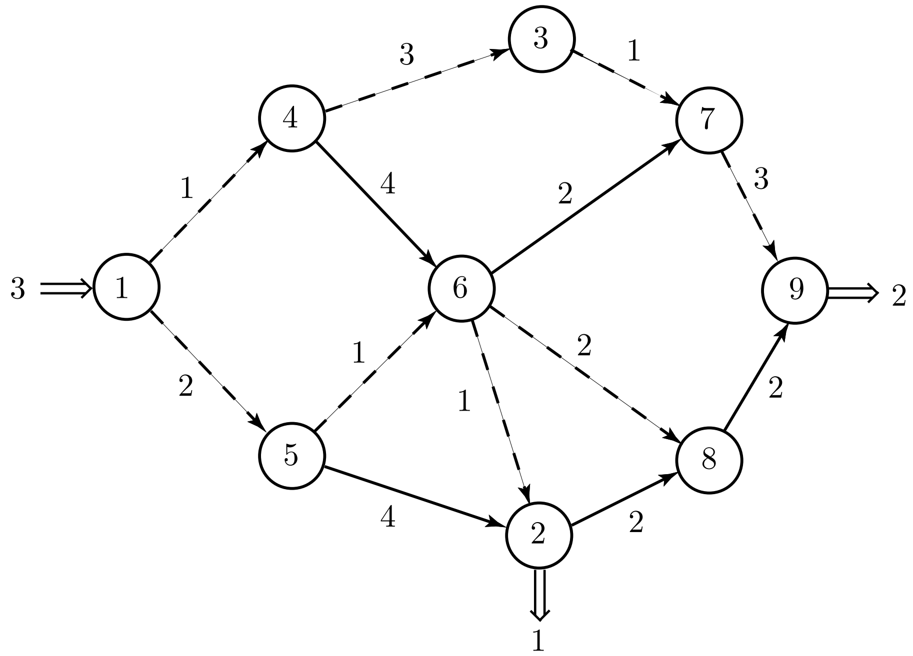
Give an account of the ellipsoidal algorithm for the problem of deciding whether or not P is empty. Describe the inputs to the algorithm and its main steps. You need not derive any detailed formulae, but you should explain enough so that the role of the above result is clear.

3 Define the uncapacitated minimum cost network flow problem.

Define the Lagrangian for this problem. Derive dual feasibility and complementary slackness conditions that can be used to identify an optimal flow. Decribe how a spanning tree can be used to calculate a feasible choice of dual variables.

In the network below, the label next to each arc is the cost per unit flow on that arc, c_{ij}. A flow of b_i enters at node i, where $b_1 = 3$, $b_2 = -1$, $b_9 = -2$ and $b_i = 0$, $i \neq 1, 2, 9$. Find the arc flows, f_{ij}, for the basic solution corresponding to the spanning tree indicated by the dashed lines.

Starting from this basic solution, explain the network simplex algorithm and show that it finds the optimal flow in one step.



4 Define the terms characteristic function, imputation, core and Shapley value payoffs as they apply to n-person coalitional games.

Consider a market in which there four participants. Player 1 values his car at 0 and wishes to sell it. Each of Players 2, 3 and 4 wishes to buy the car, and each values it at 1. Find the characteristic function, core and Shapley value payoffs.

Consider now a market in which there are $4k$ participants, of which each of k participants has a car to sell and each of the other $3k$ participants is a potential buyer of one car. All cars are identical, of value 0 to sellers and value 1 to buyers. Show that as $k \to \infty$ the Shapley value payoff of a seller tends to 1 and the Shapley value payoff of a buyer tends to 0.

5 An instance of the ΔTSP *decision problem* is an undirected graph (with all possible edges present), a nonnegative integer cost $c_{ij} = c_{ji}$ for each edge $\{i, j\}$ and a nonnegative integer L. Edge costs are required to satisfy the triangle inequality. The question is whether there is a tour whose cost is no greater than L. Show that this problem is in the complexity class $\mathcal{NP}$.

The ΔTSP *evaluation problem* is defined on the same instances (but omitting L), and the problem is to find the length of the shortest tour. Show that there exists a polynomial time algorithm for this problem if and only if there exists a polynomial time algorithm for the decision problem.

An instance of HCP is a graph G (with only some of the possible edges present). The question is whether there exists a tour that visits each vertex exactly once (a Hamiltonian circuit). Show that if HCP is $\mathcal{NP}$-complete then the ΔTSP decision problem is also $\mathcal{NP}$-complete.

Given the same data as a ΔTSP evaluation problem, the ΔTSP *optimization problem* is to find a minimum length tour; the MST optimization problem is to find a minimum spanning tree. Show that if there exists a polynomial time algorithm for the MST optimization problem then there also exists a polynomial time 1-approximation algorithm for the ΔTSP optimization problem, such that it produces a tour no more than twice the length of the minimal length tour.

6 Define what is meant by an equilibrium pair for a non-zero-sum two-person game.

State conditions under which at least one equilibrium pair is guaranteed to exist.

Two friends have different preferences for composers. Without consulting one another, they must each book for one of three possible concerts. They are pleased if they happen to book for the same concert. This is modelled by a game with the following payoff matrix.

$$\begin{array}{c c}
 & \begin{array}{ccc} \text{Bach} & \text{Mozart} & \text{Schubert} \end{array} \\
\begin{array}{c} \text{Bach} \\ \text{Mozart} \\ \text{Schubert} \end{array} &
\left(\begin{array}{ccc} (4,2) & (1,1) & (0,0) \\ (0,0) & (2,4) & (0,0) \\ (0,0) & (0,0) & (3,3) \end{array}\right)
\end{array}$$

Find all the equilibrium pairs.

UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday 4 June, 2003 1:30 to 4:30

PAPER 37

Mathematics of Operational Research

Attempt **FOUR** *questions.*

There are **six** *questions.*

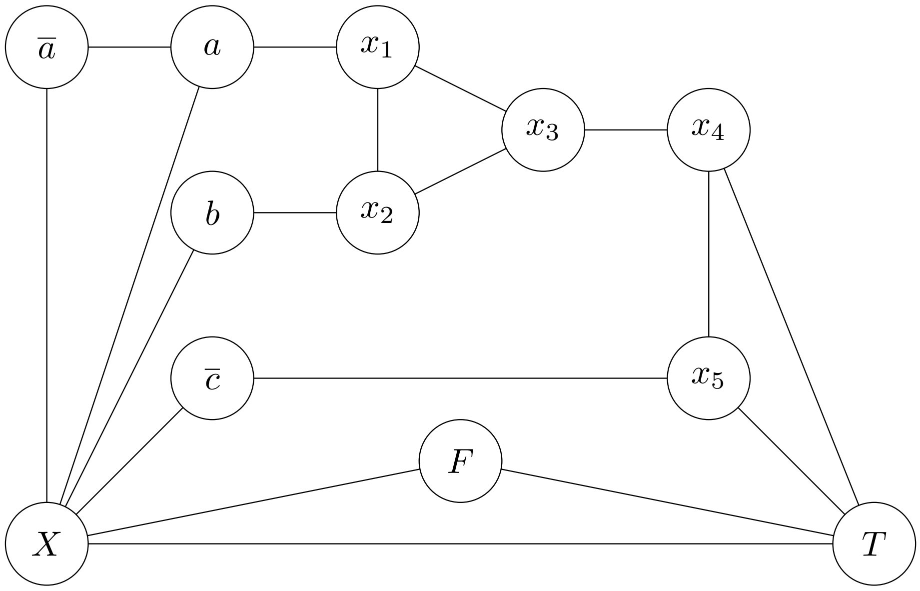
The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Explain the meanings of $\mathcal{NP}$ and $\mathcal{NP}$-complete.

A boolean formula in 3-conjunctive normal form is a conjunction (and) of several clauses, each of which is the disjunction (or) of exactly 3 literals, each of which is either a variable or its negation. An example is '(a or b or $\bar{c}$) and ($\bar{a}$ or b or c)', where $\bar{a}$ denotes the negation of a. In 3-SAT we are given such a formula and asked to say whether there exists an assignment of the variables (to 'true' or 'false') such that the formula is true. Show that 3-SAT is in $\mathcal{NP}$.

In 3-COLOURABILITY we are given a graph as input and asked to decide whether it has a 3-colouring. That is, can we colour the nodes with 3 different colours so that every two nodes that have an edge between them are of different colours? Consider the following statement: *there exists a 3-colouring of the following graph if and only if at least one of the nodes a, b or $\bar{c}$ is coloured the same colour as node T.* Prove the 'only if' part.



Given that the 'if' part is also true and that 3-SAT is $\mathcal{NP}$-complete, show that 3-COLOURABILITY is $\mathcal{NP}$-complete.

2 Explain what is meant by saying that a polyhedron P is full-dimensional.

Let $P = \{x \in \mathbb{R}^n : Ax \geqslant b\}$ and assume that A and b have integer entries which are bounded in absolute value by U. Let

$$\epsilon = \frac{1}{2(n+1)} \left[(n+1)U\right]^{-(n+1)}, \quad P_\epsilon = \{x \in \mathbb{R}^n : Ax \geqslant b - \epsilon e\}$$

where $e^\top = (1, 1, \ldots, 1)$. Show that if P is non-empty, then P_ϵ is full-dimensional.

Give a brief account of the ellipsoidal algorithm for the problem of deciding whether or not P is empty. Describe the inputs to the algorithm and its main steps. You need not derive any detailed formulae, but you should explain enough so that the role of the above result is clear.

3 Employees 1, 2, 3, 4, are to be assigned to the jobs 1, 2, 3, 4, one person per job. The cost of assigning person i to job j is a_{ij}, where these are elements of the matrix

$$\begin{pmatrix} 11 & 12 & 18 & 40 \\ 14 & 15 & 13 & 22 \\ 11 & 17 & 19 & 23 \\ 17 & 14 & 20 & 28 \end{pmatrix}$$

Use the branch and bound method to solve this problem. You should take as a partial solution an assignment of persons $1, \ldots, k$ to k different jobs, $k \leqslant 3$, and use as a lower bound for this partial solution the cost of all the assignments made so far, plus the sum of the least costs with which each of the remaining unassigned jobs could be assigned to one of the persons $k+1, \ldots, 4$ (without requiring each of these jobs to be assigned to a distinct person). Start with the four partial solutions in which person 1 is assigned to job 1, 2, 3 or 4.

Explain how assignment problems can be used with a branch and bound approach to solve the travelling salesman problem.

4 Give an account of Nash's bargaining game, bargaining axioms and arbitration procedure.

Consider the two person non-zero sum game with payoffs

$$\begin{array}{cc} & \begin{array}{cc} II_1 & II_2 \end{array} \\ \begin{array}{c} I_1 \\ I_2 \end{array} & \begin{pmatrix} (2,4) & (8,2) \\ (4,5) & (2,3) \end{pmatrix} \end{array}$$

Find the Nash bargaining solution when the status quo point is taken as the maximin point.

5 Consider an n-person game in which players have strategies $p_1, \ldots, p_n$, each of which may be a mixed strategy. A strategy p_i^* for player i is said to be *dominant* if regardless of what his opponents do it gives him at least as good a payoff as any other strategy he might adopt. Show that if $p_1^*, \ldots, p_n^*$ are dominant strategies for players $1, \ldots, n$ respectively, then $p_1^*, \ldots, p_n^*$ is a Nash equilibrium.

Consider a sealed-bid auction in which the bidders have symmetric independent private values. The winner is the highest bidder and he pays the amount of the second highest bid. Show that a dominant strategy for bidder i is to bid his true valuation.

Suppose, instead, that the winner pays the amount of his own bid. State, or prove the nonexistence of: (a) a dominant strategy for bidder i; (b) a Nash equilibrium.

6 Define the notion of an evolutionary stable strategy (ESS) and derive necessary and sufficient conditions for a strategy $\mathbf{x}^*$ to be evolutionary stable. Use the notation that $e(\mathbf{x}, \mathbf{y})$ is the payoff to a player who uses strategy $\mathbf{x}$ against an opponent who uses strategy $\mathbf{y}$.

Consider the Hawk vs Dove game with payoffs (to the row player) of

$$\begin{array}{cc} & \begin{array}{cc} \text{Hawk} & \text{Dove} \end{array} \\ \begin{array}{c} \text{Hawk} \\ \text{Dove} \end{array} & \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} \end{array}$$

Show that the mixed strategy $(\frac{1}{2}, \frac{1}{2})$ is evolutionary stable.

UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday 1 June, 2004 13:30 to 16:30

PAPER 33

Mathematics of Operational Research

Attempt **FOUR questions.**

There are **six questions in total.**

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Consider the linear program

$$P: \quad \text{maximize} c^\top x\,, \quad Ax \leqslant b \quad \text{and} \quad x \geqslant 0.$$

Here $c, x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ and A is $m \times n$. Derive the dual problem, D.

Prove or provide a counterexample to each of the following.

(a) If P is unbounded then D is infeasible.

(b) If P is infeasible then D is unbounded.

The following is the final simplex algorithm tableau for a linear programming problem P, in which $n = m = 2$. Find all the optimal solutions to both the primal and the dual problems.

What was the original primal problem?

x_1	x_2	z_1	z_2	
0	1	$\frac{1}{10}$	$\frac{1}{10}$	1
1	0	$\frac{1}{20}$	$\frac{3}{20}$	2
0	0	0	$-\frac{1}{2}$	-3

2 Given a graph (N, A) with flow capacities on the arcs, and nodes $s, s' \in N$ it is desired to maximize the flow from s to s'. Assuming the theorem that min-cut equals max-flow, prove that the Ford-Fulkerson algorithm solves this problem.

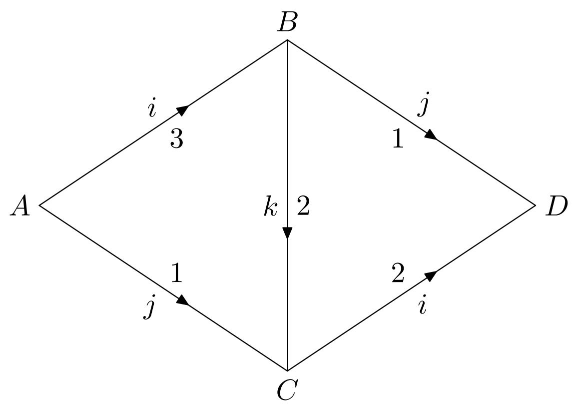
A total of n teams play in a football league. Thus far in the season team i has won w_i games. Teams i and j are still to play one another in g_{ij} games. We want to know if it is possible for team n to end the season having won more games than any other team. Explain how this problem can be addressed via a maximum flow problem in which, for each $i \neq j$, $i, j < n$, a node s is connected to a node r_{ij} with an arc of capacity g_{ij}, each node r_{ij} is also connected to nodes t_i and t_j with arcs of infinite capacity, and each node t_i is connected to a node s' with an arc of capacity $w^* - w_i - 1$ and $w^* = w_n + \sum_{i<n} g_{in}$.

Suppose that, regardless of the size of n, every parameter of the problem can be represented in no more than k bits. Is the problem in $\mathcal{P}$?

3 Write an essay on the minimum spanning tree problem. Carefully explain what is meant by complexity classes $\mathcal{P}$ and $\mathcal{NP}$ and prove that the minimum spanning tree problem belongs to both of these.

4 Define the terms *characteristic function* and *core* of a cooperative game.

In the network below the five directed links between pairs of nodes A, B, C, D are owned by three communications companies i, j and k, as marked. The maximum number of units of data flow that can be carried on each link is shown by an integer beside the link. Customers will pay £6000 per unit of data flow from A to D (irrespective of the path it takes) and £4000 per unit of data flow from B to C. By formulating an appropriate minimum cost network flow problem, show that the maximum possible revenue is £22000.



The companies are trying to reach an agreement about how much data traffic should be carried and how the resulting revenue should be shared. Specify by a set of constraints all the ways that they could share the revenue of £22000 such that no subset of companies would have any incentive to prefer operating without the others.

Viewing the above as a cooperative game, find the nucleolus.

5 40 jobs are to be processed sequentially on a single machine in some order. The processing time of job i is t_i. If job i is the first job to be processed on the machine then a time s_i will be required to set up the machine. If job j immediately follows job i then a time s_{ij} will be required to change tools on the machine. Show how the problem of finding the schedule that minimizes the time to complete all the jobs can be formulated as a 0–1 linear programming problem.

Describe how you could address this problem by

(a) a branch and bound algorithm, and the solution of a sequence of assignment problems;

(b) a simulated annealing algorithm, using a 2-opt heuristic.

Suppose now that three machines are available to work on the jobs in parallel. Suggest a strategy for tackling this problem.

6 Consider a n-person game in which each player has a finite number of pure strategies. Define what is meant by a *Nash equilibrium*.

Show that a Nash equilibrium in pure strategies may not exist.

50000 students are applying to 50 universities. The students can be ranked $1, \dots, 50000$ and each student knows his own rank. Each student is permitted to apply to exactly one university. Once all applications are in, each university accepts up to 1000 students, choosing those with the greatest ranks amongst all those who apply. An accepted student obtains a benefit equal to the average rank of those students who are accepted at his university. A student who is not accepted obtains benefit 0. Show that there are exactly 50! different Nash equilibria in pure strategies.

Suppose now that we allow mixed strategies. Does this lead to 0, 1, or more than 1, further Nash equilibrium?

UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 3 June, 2005 9 to 12

PAPER 40

MATHEMATICS OF OPERATIONAL RESEARCH

*Attempt **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Use the dual simplex algorithm to solve the problem:

$$\text{minimize} \quad 2x_1 + 15x_2 + 18x_3$$

subject to

$$\begin{array}{rcrcrcr}
-x_1 & + & 2x_2 & - & 6x_3 & \leqslant & -10 \\
 & & x_2 & + & 2x_3 & \leqslant & 6 \\
2x_1 & & & + & 10x_3 & \leqslant & 19 \\
-x_1 & + & x_2 & & & \leqslant & -2
\end{array}$$

$$x_1, x_2, x_3 \geqslant 0\,.$$

Now use Gomory's cutting plane method to solve the problem when x_1, x_2, x_3 must be integers.

2 Let FP denote the feasibility problem: 'Is the set $P = \{x : Ax \geqslant b,\, x \in \mathbb{R}^n\}$ nonempty?' Here A is a $m \times n$ matrix, where $m \geqslant n$, and the components of A and b are integers with absolute values no more than U. How many bits are needed to state an instance of FP?

Show that if P is nonempty then there exists $x \in P$ such that each component of x can be written as the quotient of two integers, each of which is in absolute value no more than $(nU)^n$.

Deduce that FP is in complexity class $\mathcal{NP}$.

Assuming that the ellipsoid algorithm can solve FP in polynomial time, prove that there exists a polynomial-time algorithm for the problem: minimize $c^\top x$, $Ax \geqslant b$.

3 State and prove Nash's theorem concerning the existence of an equilibrium in a two-person non-zero-sum game. You may assume the Brouwer Fixed Point Theorem.

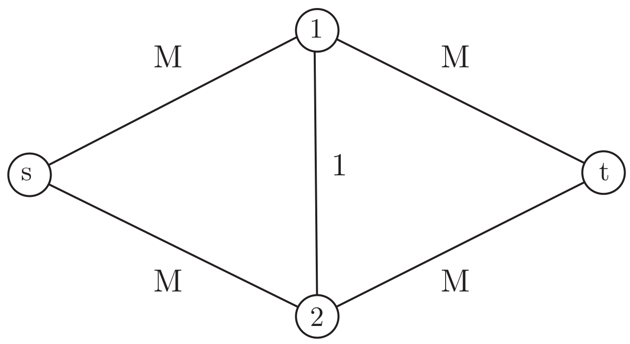
4 State a formula for the Shapley values of a coalitional game. What axioms do they satisfy?

Show that if each player receives a payoff equal to his Shapley value then it is true to say: 'The payoff I lose if you leave the game is equal to the payoff you lose if I leave the game.'

Suppose agent i knows about a set of books B_i. If a set of agents S pool what they know then their payoff is the number of books about which they collectively know, i.e., $\left|\bigcup_{i\in S} B_i\right|$. Show that the game is superadditive and the core is nonempty only if the sets $B_1, \ldots, B_n$ are disjoint.

Show that agent i has Shapley value $x_i = \sum_{b\in B_i} \left|\{k : b \in B_k\}\right|^{-1}$.

5 Consider the undirected graph below, with integer-valued capacities marked on its edges. It is desired to find the maximum flow between s and t. Show that, depending on choices made, the Ford-Fulkerson algorithm might take as few as 2 or as many as $M + 1$ steps to terminate.



Suppose that in a certain undirected graph G with integer-valued edge capacities (c_{ij}) the maximum possible flow between nodes s and t is f^*. Let (x_{ij}) be a feasible flow that sends flow of f from s to t, where x_{ij} is the flow from i to j along edge $\{i, j\}$. Let the 'residual graph' be obtained by supposing edges are directed and the capacities are set to $c'_{ij} = c_{ij} - x_{ij} + x_{ji}$. By using the fact that the minimum cut equals maximum flow show that the maximal flow possible between s and t in the residual graph is $f' = f^* - f$.

Let m be the number of edges of G. Let U be the set of nodes in the residual graph that can be connected to s by a path of capacity of at least f'/m. Show that $t \in U$.

A modified Ford-Fulkerson algorithm adds the rule that whenever flow might be increased on more than one path from s to t then we choose a path on which the greatest increase can be made. Show that after k steps of this algorithm the maximal flow possible from s to t in the residual graph is no more than $(1 - 1/m)^k f^*$.

Hence prove that this algorithm terminates in $O\big(m \log(f^*)\big)$ steps.

6 In a certain a sealed-bid auction bidders compete for a single item. The winner is the highest bidder and he pays the second highest bid. Show that it is a Nash equilibrium for each bidder to bid his valuation.

Explain what is meant by an auction with symmetric independent private values (SIPV).

Suppose a SIPV auction has n bidders whose valuations are uniformly distributed on $[0, 1]$. Show that if the winner must pay his own bid then it is not a Nash equilibrium for bidders to bid their valuations, but that it is a Nash equilibrium for them to bid $(n-1)/n$ times their valuations.

END OF PAPER

UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 1 June 2006 1.30 to 4.30

PAPER 38

MATHEMATICS OF OPERATIONAL RESEARCH

*Attempt **FOUR** questions.*

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STATIONERY REQUIREMENTS
Cover sheet
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Script paper

SPECIAL REQUIREMENTS
None

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1 Consider the linear programming problem:

$$\begin{array}{lcl} \text{maximize } 4x_1 + 2x_2 + x_3 & & \\ \text{subject to} & & \\ x_1 & \leqslant & 5 \\ 4x_1 + x_2 & \leqslant & 25 \\ 8x_1 + 4x_2 + x_3 & \leqslant & 125 \end{array}$$

and $x_1, x_2, x_3 \geqslant 0$. Solve this with the simplex algorithm, starting from $x = (0, 0, 0)$, and using the rule that whenever there is a choice as to which variable should next enter the basis it should selected as the one that produces the greatest increase in the objective function per unit increase in that variable.

Is there a pivot selection rule under which the problem would have been solved more quickly?

Discuss the worst-case running time of the simplex algorithm.

2 Explain what is meant by the minimum-cost flow problem.

A project of n tasks is to be completed as quickly as possible. We may work on more than one task at the same time, but we are subject to precedence constraints expressed in the matrix $a = (a_{ij})$, such that if $a_{ij} = 1$ we may not start task j until task i is complete; otherwise $a_{ij} = 0$ and there is no precedent constraint between i and j. Task i has processing time τ_i, $i = 1, \ldots, n$. Let t_i be the earliest time at which task i can be started. Formulate as a linear program the problem of minimizing $t_{n+1} - t_0$, where t_0 and t_{n+1} are the times at which the project starts and finishes.

Show that the dual LP can be expressed as an uncapacitated minimum cost flow problem on a graph (N, A) where $(i, j) \in A$ if and only if $a_{ij} = 1$ and the cost on arc (i, j) is $c_{ij} = -\tau_i$.

Illustrate an algorithm that can be used to solve any minimum cost flow problem by applying it to the project of 6 tasks defined by the following data. Start your explanation at a basic feasible solution corresponding to the tree with arcs $\{(0,1), (2,4), (4,5), (5,7), (4,6), (1,3), (3,5)\}$.

$$a = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \tau = (3, 4, 5, 6, 4, 3).$$

3 Explain in terms of the theory of computational complexity what it means to say that a problem Π is no harder than another problem Π'.

Let A be the $m \times n$ payoff matrix for a two-person zero-sum game in which players 1 and 2 have m and n pure strategies respectively. Given A and a number V, let Π be the problem of determining whether the value of the game equals V. Sketch an argument to show that if $\mathcal{P} \neq \mathcal{NP}$ then Π is not $\mathcal{NP}$-hard.

In an instance of the Subset Cover Problem, SCP, we are given subsets $S_1, \ldots, S_m$ of $S = \{1, \ldots, n\}$, and a number $k < m$, and ask 'Are there k of these subsets whose union is S?' Consider the non-zero-sum game in which if player 1 plays pure strategy $i \in \{1, \ldots, m\}$ and player 2 plays pure strategy $j \in \{0, 1, \ldots, n\}$, the payoff is

$$((e_1(i,j), e_2(i,j)) = \begin{cases} (1,1) & j = 0, \\ (1,0) & j \geqslant 1 \text{ and } j \in S_i, \\ (0, \frac{k}{k-1}) & j \geqslant 1 \text{ and } j \notin S_i. \end{cases}$$

Show that if and only if the answer to the SCP instance is 'Yes' does there exist an equilibrium in which player 1 randomizes with positive probabilities over just k of his pure strategies.

Comment on the difficulty of computing all equilibria of a two-person non-zero-sum game.

4 List the assumptions of a symmetric independent private values (SIPV) auction.

State the revenue equivalence theorem.

Let $e(p)$ denote the minimal expected payment that a bidder can make if he wishes to win an SIPV auction with probability p. Suppose that when a bidder with valuation v seeks to maximize his expected profit he does so by choosing $p = p(v)$ as a stationary point of his expected profit function. Show that

$$\left.\frac{de(p)}{dp}\right|_{p=p(v)} = v, \quad \frac{de(p(v))}{dv} = v\frac{dp}{dv}, \quad \text{and} \quad e(p(v)) = vp(v) - \int_0^v p(w)\, dw\,.$$

Consider a 'lowest-price auction' amongst n bidders in which the highest bidder wins but he pays only the lowest bid. Assume that bidders' valuations are independent and uniformly distributed on $[0, 1]$. Show that, at the equilibrium, the seller's expected revenue is $(n-1)/(n+1)$.

Suppose that when $n = 3$ there exists a contant A such that it is optimal in equilibrium for a bidder with valuation v to bid Av. Find A.

5 Define the meaning of an equilibrium in a multi-person game.

In a small town there are just 3 residents. A proposition regarding taxes is favoured by residents B and C, but opposed by A. It will be passed in a ballot if and only if it receives more votes in favour than against. Each resident has a cost of going to the polls to vote of $3c$. If the proposition passes, each of B and C will gain by $4c$ and A will lose $8c$. Suppose A decides to go to the polls with probability α and each of B and C go independently with probability β.

Find a condition that must be satisfied by β if there is to be an equilibrium with $0 < \alpha < 1$.

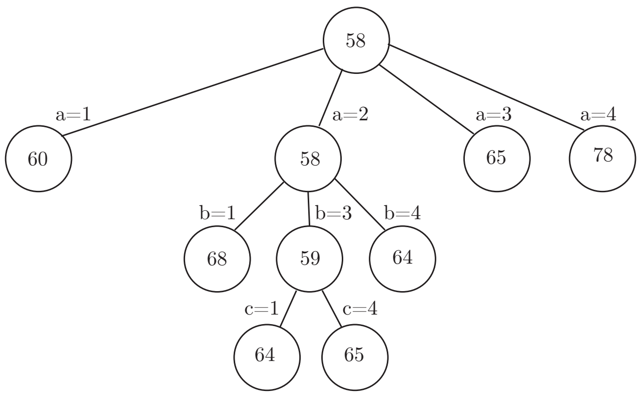
Show that there is an equilibrium of $\alpha = 1$, $\beta = 3/4$.

Is this the only equilibrium?

6 Describe the methodology of branch and bound algorithms.

Consider an assignment problem in which four machines, a, b, c, d are to be assigned to four tasks 1, 2, 3, 4 at minimal cost. The costs of assigning machines to tasks are given in the matrix below. At the first stage of a branch and bound algorithm there are four branches, in which machine a is assigned to either task 1, 2, 3 or 4. If machine a is assigned to task 2, then a lower bound is computed by adding this cost (12) to the minimum costs with which each of the unassigned jobs, 1, 3, 4, can be assigned to some unassigned machine, e.g., 11, 13, and 22, respectively, for a total lower bound of 58. Branches from this node, are those for which machine b is assigned to jobs 1, 3, or 4. In the branch in which machine b is assigned to task 1 we would have a lower bound of $12+14+19+23 = 68$. Using a branching rule that branches on the node with least lower bound, complete the branch and bound algorithm that is begun in the figure below, and find the optimal assignment.

	1	2	3	4
a	11	12	18	40
b	14	15	13	22
c	11	17	19	23
d	17	14	20	28



END OF PAPER

UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 4 June 2007 9.00 to 12.00

PAPER 40

MATHEMATICS OF OPERATIONAL RESEARCH

*Attempt **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Use the simplex algorithm to solve the problem

$$\text{maximize}\, x_1 + x_2 \text{ subject to}$$

$$\begin{aligned} x_1 - x_2 &\geqslant -2 \\ x_1 + 2\,x_2 &\leqslant 8 \\ 2\,x_1 + x_2 &\leqslant 8 \\ x_1,\ x_2 &\geqslant 0 \end{aligned}$$

Suppose we now add to this problem the constraint that x_1 and x_2 must be integers. Use Gomory's cutting plane method and the dual simplex algorithm to find all the optimal solutions. Carefully explain the rationale for any new constraints that you introduce.

2

Discuss the representation of the travelling salesman problem as an integer linear programming problem. Show that if there are n cities and all intercity distances are integers that are at most 2^n, then the travelling salesman problem can be expressed as an integer linear program with size of $O(n^3)$ bits.

Define the class of decision problems $\mathcal{NP}$-complete.

Show that if travelling salesman decision problems are $\mathcal{NP}$-complete then integer linear programming decision problems are also $\mathcal{NP}$-complete.

3

Consider the problem '*Is the polyhedron* $P = \{x \in \mathbb{R}^n : Ax \geqslant b\}$ *nonempty?*' Suppose it is known that if P is nonempty then it is contained within the ellipsoid $E = E(z, D) = \{x : (x - z)^\top D^{-1}(x - z) \leqslant 1\}$. Show that if $z \notin P$ and P is non-empty then P must be contained in the intersection of E and some half space H.

Suppose D is the $n \times n$ identity matrix, $H = \{x : x_1 \geqslant 0\}$, and $e_1 = (1, 0, \ldots, 0)\}$. Given that

$$E' = E\left(\frac{e_1}{n+1}, \frac{n^2}{n^2-1}\left(I - \frac{2}{n+1}\; e_1 e_1^\top\right)\right)$$

is an ellipsoid containing $E \cap H$, show that the volume of E' is less than $e^{-1/(2(n+1))}$ times the volume of E. You may use the fact that the volume of $E\,(z, D)$ is proportional to $\sqrt{\det(D)}$.

Briefly discuss the importance of the above result in constructing a polynomial time algorithm for linear programming.

4

Explain what is meant by the characteristic function of a coalitional game. Why is the characteristic function always a superadditive function?

A company is bankrupt and owes three creditors amounts of money $c_1 = 4$, $c_2 = 6$, $c_3 = 9$ (in 10000 s of pounds). Unfortunately it has assets of only $a = 15$. A liquidator proposes to divide these assets so that creditors 1, 2 and 3 receive $x_1 = 8/3$, $x_2 = 14/3$ and $x_3 = 23/3$, respectively.

For the characteristic function defined by

$$v(S) = \max\left\{ 0\,, \quad a - \sum_{i \notin S} c_i \right\}, \qquad S \subseteq \{1, 2, 3\}\,,$$

and $x = (x_1, x_2, x_3)$, determine which of the following are true. Your answer should include definitions of 'Pareto optimal', 'Shapley value', 'nucleolus' and 'core'.

(a) x is Pareto optimal.

(b) x_1, x_2, x_3 are Shapley values.

(c) x is the nucleolus.

(d) x is in the core.

5

A seller is preparing to sell a used car. He knows that there are just 2 potential buyers. He considers 3 methods of selling.

(a) He offers the car at price p and waits to see if anyone buys it.

(b) He conducts an oral ascending price auction, selling the car to the highest bidder, who must pay his bid.

(c) He modifies (b) by accepting no bid less than his reserve price r. A buyer can win only if he is the highest bidder and bids more than r. The winner (if any) pays his bid.

Suppose that the buyers have independent private valuations of v_1 and v_2, where *a priori* these can be modelled as independent uniform random variables on $[0, 1]$ (measuring, perhaps, fractions of 2000 pounds). Assuming that the seller chooses p and r optimally, determine his expected revenue under each selling method. Which is best?

Explain why the fact that the expected revenue differs under (b) and (c) does not contradict the revenue equivalence theorem for SIPV auctions.

6

Given a set of n items, with positive weights $\{w_1, w_2, \ldots, w_n\}$, we wish to find the least y such that the items can be placed in 2 bins and the total weight in each bin is no more than y.

(a) Formulate the problem as an integer linear programming problem.

(b) Define the notion of an ϵ–approximation algorithm for a minimization problem.

Let $s = \sum_i w_i$ and $w_{\max} = \max_i w_i$. Consider an algorithm in which we place the items in the bins in the order $1, 2, \ldots, n$, always placing an item in a bin that presently contains the least total weight. Show that just before it receives item j, the bin which receives item j contains total weight that is no more than $(1/2)(s - w_j)$ (for any j). Deduce that the algorithm achieves $y \leqslant s/2 + w_{\max}/2$. Hence determine the least ϵ for which this algorithm can be claimed to be an ϵ–approximation algorithm.

Hint: consider an example of 3 items, with weights 1, 1 and 2.

(c) Using the bin-packing problem above to illustrate your ideas, discuss other means of obtaining approximate solutions to combinatorial optimization problems.

END OF PAPER

UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 2 June 2008 1.30 to 4.30

PAPER 38

MATHEMATICS OF OPERATIONAL RESEARCH

Attempt no more than ***FOUR*** *questions.*

There are ***SIX*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
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SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let P be the linear programming problem: maximize $\{c^\top x \,:\, Ax \leqslant b,\ x \geqslant 0\}$, where $x, c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ and A is $m \times n$. What is its dual, D?

Explain why the following are true.

(a) If P is feasible then D is bounded.

(b) If P is feasible and bounded then D is feasible and bounded.

Suppose that the polytope $\Pi = \{x \,:\, Ax \leqslant b,\ x \geqslant 0\}$ is empty. Show that there exists some $\lambda \geqslant 0$ such that $\lambda^\top A \geqslant 0^\top$ and $\lambda^\top b = -1$.

Let $\Pi(\epsilon) = \{x \,:\, Ax \leqslant b + \epsilon e,\ x \geqslant 0\}$, where $e^\top = (1, \ldots, 1)$ and $\epsilon \geqslant 0$. Given that $\Pi = \Pi(0)$ is empty, and λ is as above, show that $\Pi(\epsilon)$ is empty for all ϵ such that $0 \leqslant \epsilon < 1/\sum_i \lambda_i$.

Explain the relevance of this result to the theory of the ellipsoid algorithm.

2 The Klee-Minty polytope in $\mathbb{R}^3$ is the intersection of the six halfspaces on which $x = (x_1, x_2, x_3)$ satisfies the following constraints, for given ϵ, $0 < \epsilon < 1/2$:

$$
\begin{aligned}
&x_1 \geqslant 0\,,\\
&x_1 \leqslant 1\,,\\
&x_2 \geqslant \epsilon x_1\,,\\
&x_2 \leqslant 1 - \epsilon x_1\,,\\
&x_3 \geqslant \epsilon x_2\,,\\
&x_3 \leqslant 1 - \epsilon x_2\,.
\end{aligned}
$$

This polytope, P, has six facets, which are respectively indexed as $1, 2, \ldots, 6$, as they lie on a boundary of each of the above six halfspaces. The vertex at the intersection of the first, third and fifth facets is $v_0 = (0, 0, 0)$. Bland's pivot rule says that at each successive step of the simplex algorithm the solution should move from the current vertex of the feasible set to an adjacent vertex; the objective function value should increase, and if there is more than one adjacent vertex at which that value increases then we should pick the one that we move to along the edge that is leaving the facet of smallest index. It is desired to maximize x_3 over P. Show, using a picture, the steps taken by the simplex algorithm under Bland's rule, starting from v_0.

Discuss the worst-case running time of the simplex algorithm under Bland's rule.

Show that there are examples of linear programs, in decision variables $x \in \mathbb{R}^n$, and with $2n$ constraints, in which it takes at least n pivots to move from an initial solution to the optimum, no matter how the pivots are chosen.

An alternative to Bland's pivot rule is Dantzig's rule, in which we are to move to an adjacent vertex along an edge for which the rate of increase in the objective function is greatest. Describe how to modify P to show that there is a similar example for which Dantzig's rule is inefficient.

3 Describe the minimum cost flow problem.

Explain how one can use the Lagrangian sufficiency theorem to identify an optimal solution by means of node numbers.

A network has nodes $N = \{1, 2, 3, 4\}$ and directed arcs $A = \{(1,2), (2,3), (3,4), (4,1)\}$. Nodes 1, 3 and 4 are sources, for flow amounts 1 each. Node 2 is a sink, for flow of amount 3. The minimum permitted flows on arcs $(1,2)$, $(2,3)$, $(3,4)$, $(4,1)$ are 3, 2, 2, 0, respectively; the maximum permitted flows are 8, 5, 5, 8, respectively. The costs per unit flow on these arcs are c_{12}, c_{23}, c_{34}, c_{41}, respectively. Show that there is a feasible solution in which arc $(2,3)$ carries flow of 2.

Derive a condition in terms of c_{12}, c_{23}, c_{34}, c_{41} under which this is the minimum cost flow.

Use the network simplex algorithm to find the minimum cost flow for all possible values of the four cost variables $\{c_{ij}\}$, $c_{ij} \in (-\infty, \infty)$.

Determine the numbers of (i) basic solutions, and (ii) basic feasible solutions to this problem.

4 Let $G = (V, E)$ be an undirected graph. Edge e has weight w_e and the edge weights are distinct, say $w_1 < \cdots < w_{|E|}$. Let S be a nonempty proper subset of V, and let edge $e = (u, v)$ be a edge of least weight that has one end in S and the other end in $V \setminus S$. Prove that every minimum spanning tree must contain the edge e.

Use the above result to prove that a minimum spanning tree can be found both by Prim's algorithm (which you should state), and by Kruskal's algorithm (in which we build a spanning tree by successively considering edges in order of increasing edge weight, inserting an edge if this does not create a cycle).

Is the minimum spanning tree unique?

Let C be a cycle in G, and let the edge $e = (v, w)$ be the edge in C of maximum weight. Prove that no minimum spanning tree can contain e. Use this to prove that the minimum spanning tree problem can also be solved by a 'reverse Kruskal's algorithm', in which we start with the full graph (V, E) and then successively consider edges in order of decreasing weight, deleting an edge if this does not disconnect the graph.

5 Describe how to formulate the decision travelling salesman problem (TSP) as a 0–1 integer linear programming problem.

Carefully explain what it means to say that decision TSP is $\mathcal{NP}$-complete.

In 'decision Max-TSP' the aim is to decide where there is a tour of length greater than some given L. Given that TSP is $\mathcal{NP}$-complete, show that Max-TSP is also $\mathcal{NP}$-complete.

Find a polynomial time 1/2-approximation algorithm for a Max-TSP optimization problem, that is, an algorithm that produces a tour with length no less than 1/2 the optimum. Hint: obtain an upper bound for the Max-TSP problem from the solution to an assignment problem, and then modify this solution so that a single tour is obtained.

6 Define the notion of a Nash equilibrium in a n-person, nonzero-sum game.

A 'symmetric game' is one in which the same strategies are available to all players and the payoff that a player obtains when playing a particular pure strategy depends only that strategy and the strategies that other players employ, not on the identities of who plays them. A symmetric equilibrium is one in which all players use the same strategy, possibly mixed. Let $e(i, p)$ be the expected payoff to a player who plays pure strategy i against opponents who independently each use the same mixed strategy, which randomizes over k pure strategies with probabilities $p = (p_1, \ldots, p_k)$. Let $e(p) = \sum_{i=1}^{k} p_i e(i, p)$. Prove that at least one symmetric equilibrium is guaranteed to exist. You may assume the Brouwer fixed point theorem.

In a 'least unique bid auction' bidders are required to make their bids from a set of values, say $\{1, 2, \ldots, k\}$ and the winner is the one who makes the least unique bid. The winner pays his bid and obtains the object, which is worth V. If there is no unique bid then there is no winner. Consider such an auction, with 3 bidders, $k = 2$ and $V = 3$. Find all the symmetric equilibria.

What is the number of nonsymmetric equilibria?

END OF PAPER

UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 29 May, 2009 1:30 pm to 4:30 pm

PAPER 35

MATHEMATICS OF OPERATIONAL RESEARCH

Attempt no more than ***FOUR*** *questions.*

There are ***SIX*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
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SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Mathematics of Operational Research

Given vectors b, c and $m \times n$ matrix $A = (a_{ij})$, and $S \subseteq \{1, \dots, m\}$ define $P(S)$ as the linear program

$$\text{maximize } c^\top x\,,$$
$$\text{such that } x \geqslant 0, \quad \text{and} \quad \sum_{j=1}^{n} a_{ij} x_j \leqslant b_i\,, \quad \text{for all } i \in S\,.$$

It is desired to find the optimal value of $P(\{1, \dots, k\})$ for all $k \in \{1, 2, \dots, m\}$ for which there exists a feasible solution. Starting with $P(\{1\})$, and then proceeding from its solution, use the dual simplex algorithm to solve this problem for the data

$$c^\top = (1,1,1,1)\,, \quad A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 2 & 3 \\ -1 & -1 & -4 & -5 \\ 0 & 1 & 2 & 2 \\ 4 & 2 & 0 & 6 \end{pmatrix}\,, \quad b = \begin{pmatrix} 5 \\ 11 \\ -6 \\ 1 \\ 20 \end{pmatrix}.$$

Consider the problem, having input data of arbitrary b, c, A and k, as follows:

Does there there exist a set S of size k such that $P(S)$ is feasible?

Explain why this problem is likely to be $\mathcal{NP}$-complete.

2 Mathematics of Operational Research

Suppose we are given a graph $G = (V, E)$, having n vertices and m edges. We are also given a set of edge weights $\{c_e,\ e \in E\}$, and a number k, all of these being integers in the range 1 to 100. For fixed vertices s and t, let Π be the set of all paths from vertex s to vertex t. A set of edge numbers $\{x_e,\ e \in E\}$ is said to be feasible if

$$\sum_{e \in E} c_e x_e \leqslant k,$$

$$\sum_{e \in p} x_e \geqslant 1\,, \quad \text{for all } p \in \Pi,$$

$$0 \leqslant x_e \leqslant 1\,, \quad \text{for all } e \in E.$$

It is desired to determine whether or not the feasible set, P, is not empty.

Show that the number of constraints in a general instance of this problem is not bounded by any polynomial function of n and m.

Show that if P is not empty, then it must be contained in a m dimensional sphere of volume no more than $O(200^m)$.

Explain how you could use the ellipsoid algorithm to solve this problem, so that the worst-case running time is bounded by a polynomial in n. You may state without proof facts about the algorithm.

Explain how you will solve the problem of checking (in polynomial time) whether or not the point z_t at the centre of an ellipsoid $E(z_t, D_t)$ satisfies the constraints.

3 Mathematics of Operational Research

Suppose that n facilities are to be placed at n locations, with one facility per location. A feasible solution can be associated with π, a permutation of $I = \{1, \ldots, n\}$, which dictates that facility i be assigned to location $\pi(i)$. In the quadratic assignment problem (QAP) the data are matrices $A = (a_{i,j})$ and $B = (b_{i,j})$, and we wish to find

$$\mathrm{OPT} = \min_{\pi} f(\pi)\,,$$

where

$$f(\pi) = \sum_{i} \sum_{j\,:\,j \neq i} a_{i,j} b_{\pi(i),\pi(j)}\,.$$

Assuming that the travelling salesman problem is $\mathcal{NP}$-complete, show that QAP is $\mathcal{NP}$-complete.

Define

$$\ell_{i,k} = \min_{\pi} \sum_{j\,:\,j \neq i} a_{i,j} b_{k,\pi(j)}$$

where π is a one-to-one mapping of $I - \{i\}$ to $I - \{k\}$. Let Π_k be the subset of permutations of I in which $\pi(1) = k$. Define

$$g(\Pi_k) = \min_{\pi \in \Pi_k} \sum_{i} \ell_{i,\pi(i)}\,.$$

Explain why $g(\Pi_k)$ is a lower bound on $\min_{\pi \in \Pi_k} f(\pi)$.

What algorithm could you use to find $g(\Pi_k)$?

In a QAP with $n = 3$, suppose A, B, and $L = (\ell_{i,j})$ are

$$A = \begin{pmatrix} \cdot & 2 & 7 \\ 2 & \cdot & 4 \\ 7 & 4 & \cdot \end{pmatrix}, \quad B = \begin{pmatrix} \cdot & 5 & 3 \\ 5 & \cdot & 4 \\ 3 & 4 & \cdot \end{pmatrix}, \quad L = \begin{pmatrix} 31 & 38 & 29 \\ 22 & 26 & 20 \\ 41 & 48 & 37 \end{pmatrix}.$$

Use a branch and bound algorithm to find OPT. You should initially partition the solution space into three sets, Π_1, Π_2 and Π_3.

4 Mathematics of Operational Research

(a) Describe three heuristic methods for finding good solutions to $\mathcal{NP}$-hard problems. Say how these might be applied to the following problem.

'Winner Determination Problem' (WDP): A set of bidders, $M = \{1, \ldots, m\}$, is bidding for a set of items, $N = \{1, \ldots, n\}$. For each subset $S \subseteq N$, bidder i makes a nonnegative bid, say $v_i(S)$. Having received all bids, the auctioneer wishes to partition the items into disjoint subsets, $S_1, \ldots, S_m$, which he can assign to bidders $1, \ldots, m$ respectively, to obtain

$$\mathrm{Opt}(v) = \max_{S_1,\ldots,S_m} \sum_{i \in M} v_i(S_i)\,.$$

(b) Now suppose that all v_i are increasing and submodular. This means that for all j, S and T with $j \notin S$ and $S \subseteq T \subseteq N$,

$$0 \leqslant v_i(T + \{j\}) - v_i(T) \leqslant v_i(S + \{j\}) - v_i(S)\,.$$

The following heuristic algorithm is proposed for WDP.

0. Set $S_i = \emptyset$ for all $i \in M$, and $S_0 = N$.
1. Find $i \in M$ and $j \in S_0$ such that $v_i(S_i + \{j\}) - v_i(S_i)$ is maximal. Let $S_i := S_i + \{j\}$ and $S_0 := S_0 - \{j\}$.
2. Repeat step 1 until $S_0 = \emptyset$.
3. Return the solution $S_1, \ldots, S_m$, and $A(v) = \sum_i v_i(S_i)$.

Use induction on n to prove that this is a polynomial time approximation algorithm such that $A(v) \geqslant \frac{1}{2}\,\mathrm{Opt}(v)$.

Hint. Without loss of generality, suppose that the algorithm begins by allocating item n to player m. Consider a new problem in which items $\{1, \ldots, n-1\}$ are to be allocated, with bids

$$\begin{aligned} v'_i(S) &= v_i(S)\,, \quad i \in \{1, \ldots, m-1\} \\ v'_m(S) &= v_m(S + \{n\}) - v_m(\{n\}) \end{aligned}$$

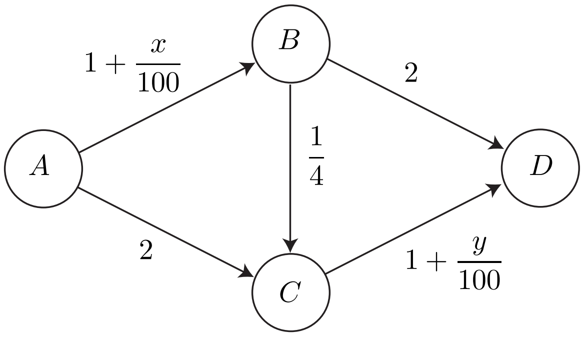
You may assume that v_i submodular implies v'_i submodular. Start by showing that $A(v) = v_n(\{m\}) + A(v')$. Then, by considering an allocation that achieves $\mathrm{Opt}(v)$ and modifying it by reallocating item n to bidder m (if it is not already so allocated), show that $\mathrm{Opt}(v') \geqslant \mathrm{Opt}(v) - 2v_m(\{n\})$.

5 Mathematics of Operational Research

Explain what is meant by a Nash equilibrium in a n person non-zero sum game.

State conditions under which a Nash equilibrium is guaranteed to exist.

In the road network below each of n players wishes to choose a route from A to D. Each player experiences a delay that is the sum of the delays on the links of his route. There is a delay of $1 + x/100$ on link AB when x players use that link, and a delay of $1 + y/100$ on link CD when y players use that link. The delays on links AC, BD and BC are 2, 2, and 1/4.



Give, in its simplest form, a set of necessary and sufficient conditions for there to be a Nash equilibrium in which n_1, n_2 and n_3 players travel on routes ABD, ACD and $ABCD$ respectively. Hence show that when $n = 100$ there is an equilibrium at $n_1 = n_2 = 25$, $n_3 = 50$.

Is this the only equilibrium in pure strategies?

Show that it would be possible for the players to follow routes that make them all better off, but that this is not a Nash equilibrium.

Find a symmetric equilibrium, i.e., one in which all players use the same strategy.

6 Mathematics of Operational Research

Explain what is meant by the characteristic function, v, of a coalitional game with a set of players $N = \{1, \ldots, n\}$.

Describe the Shapley value and say what makes it an attractive solution concept.

A game is said to be convex if its characteristic function satisfies

$$v(S + \{i\}) - v(S) \leqslant v(T + \{i\}) - v(T)\,, \quad \text{for all } S \subset T \subseteq N \text{ and } i \notin T\,.$$

Suppose (v, N) is convex and let $\phi(v, N) = (\phi_1(v, N), \ldots, \phi_n(v, N))$ be the vector of its Shapley values. Consider the game (v, T) where $T \subset N$. This is the game in which only players in subset T participate. Show that $\phi_i(v, N) \geqslant \phi_i(v, T)$ for all i.

Hence show that $\phi(v, N)$ lies in the core of the game (v, N).

A firm consists of an entrepreneur and his workers. The firm cannot operate without the entrepreneur. For any nonnegative integer k, the entrepreneur and k workers can produces profit $p(k)$ that can be shared amongst them. Assuming the firm has n workers, use the Shapley value to calculate an expression for the 'fair' wage of a worker.

What is this fair wage when $p(k) = \alpha k$?

Prove that if $p(k)$ is a convex nondecreasing function of k then the Shapley value lies within the core of this game.

END OF PAPER

UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 31 May, 2010 1:30 pm to 4:30 pm

PAPER 35

MATHEMATICS OF OPERATIONAL RESEARCH

*Attempt no more than **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

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None

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1 Consider the optimization problem

$$\begin{array}{ll} \text{Minimize} & f(x) \\ \text{subject to} & h(x) = b \\ \text{over} & x \in X\,, \end{array}$$

where $X \subset \mathbb{R}^n$ and $b \in \mathbb{R}^m$. Define the Lagrangian function for this problem and then state and prove the Lagrangian Sufficiency Theorem. Define the function ϕ by

$$\phi(b) \;=\; \inf_{x \in X} \{f(x) : h(x) = b\}\,.$$

Define the Strong Lagrangian property and show that the following are equivalent:

(a) there exists a non-vertical supporting hyperplane to ϕ at b;

(b) the problem is Strong Lagrangian.

Minimize $f = \sum v_i\, x_i^{-1}$ in $x \geqslant 0$ subject to $\sum a_i\, x_i \leqslant b$ where $a_i, v_i > 0$ for all i and $b > 0$. [In this example f is the variance of an estimate derived from a stratified sample survey subject to a cost constraint: x_i is the size of the sample for the i^{th} stratum, the a_i and v_i are measures of sampling cost and of variability for this stratum.]

Check that the change in the minimal variance f for a small change δb in available resources is $\lambda \delta b$ where λ is the Lagrange multiplier.

2

(a) A Company wishes to maximize its profit by solving the optimization problem

$$\begin{array}{lrcl} \text{Maximize} & 3\,x_1 + 4\,x_2 & & \\ \text{subject to} & x_1 + 2\,x_2 & \leqslant & 6 \\ & 2\,x_1 + \,x_2 & \leqslant & 6 \\ \text{over} & x_1 \geqslant 0\,, & & x_2 \geqslant 0\,. \end{array}$$

Use the simplex algorithm in tableau form to solve this linear problem.

(b) Subsequently the Company realises it has forgotten to include an additional constraint

$$4x_1 + 4x_2 \leqslant 15 \tag{1}$$

Use the dual simplex algorithm on the simplex tableau derived from part (a) to solve the linear program

$$\begin{array}{lrcl} \text{Maximize} & 3\,x_1 + 4\,x_2 & & \\ \text{subject to} & x_1 + 2\,x_2 & \leqslant & 6 \\ & 2\,x_1 + \,x_2 & \leqslant & 6 \\ & 4\,x_1 + 4\,x_2 & \leqslant & 15 \\ \text{over} & x_1 \geqslant 0\,, & & x_2 \geqslant 0\,. \end{array}$$

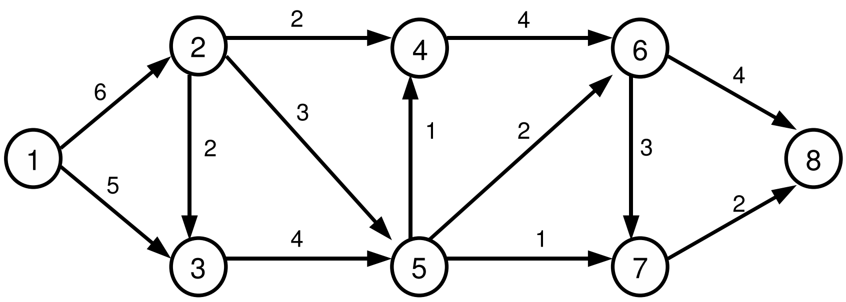
(c) Later the Company forms an agreement to sell slack in the constraint (1). Justifying any alteration made, alter the simplex tableau derived from part (b) and use the simplex algorithm to solve linear program

$$\begin{array}{lrcl} \text{Maximize} & 3\,x_1 + 4\,x_2 + y & & \\ \text{subject to} & x_1 + 2\,x_2 & \leqslant & 6 \\ & 2\,x_1 + \,x_2 & \leqslant & 6 \\ & 4\,x_1 + 4\,x_2 + y & = & 15 \\ \text{over} & x_1 \geqslant 0\,, \quad x_2 \geqslant 0\,, & & y \geqslant 0\,. \end{array}$$

3

(a) State and prove the Max-Flow Min-Cut Theorem.

(b) Use the Ford-Fulkerson Algorithm to calculate a maximum flow between a source at node 1 and a destination at node 8 in the following network.



Here the number by each arc represents the capacity of that arc. What is the min-cut of this network?

(c) Starting from the given feasible solution, minimize the cost of flows in the transportation problem given by the following tableau.

$\boxed{5}$	7 $\boxed{4}$	5 $\boxed{8}$	12
5 $\boxed{3}$	$\boxed{7}$	$\boxed{2}$	5
$\boxed{9}$	7 $\boxed{0}$	$\boxed{3}$	7
4 $\boxed{8}$	$\boxed{10}$	8 $\boxed{5}$	12
9	14	13	

[Note: In this tableau the circled numbers indicate an initial feasible set of non-zero flows, the numbers in the squares are the costs, the numbers to the right of the tableau are supplies and the numbers below are demands.]

4

Consider the Boolean formula with N clauses

$$(x_{11} \vee x_{12} \vee \ldots \vee x_{1M_1}) \wedge (x_{21} \vee x_{22} \vee \ldots \vee x_{2M_2}) \wedge \ldots \wedge (x_{N1} \vee \ldots V x_{NM_N}) \quad (1)$$

where $x_{ij} \in \{X_1, \ldots, X_K\} \cup \{\bar{X}_1, \ldots, \bar{X}_K\}$. Here $\wedge$ means "AND" and $\vee$ means "OR" and $\bar{X}$ means "NOT X".

The SAT problem considers the assignment of variables, $X_i \in \{\text{true}, \text{false}\}$, $i = 1, 2, \ldots, K$ such that (1) is true.

The MAX-SAT problem considers the assignment of variables such that the maximum number of clauses in (1) are true.

Express the SAT problem as an integer linear program.

Express the MAX-SAT problem as an integer linear program.

Consider the following approximation algorithm for MAX-SAT.

Greedy: Pick the variable $z \in \{X_1, \ldots, X_K\} \cup \{\bar{X}_1, \ldots, \bar{X}_K\}$ that occurs in the largest number of clauses in (1). Set z true and $\bar{z}$ false. This reduces formula (1) to an expression on $K-1$ variables. Repeat until no variables remain.

Show that Greedy is a $\frac{1}{2}$-approximation of MAX-SAT.

[*Recall:* Algorithm H with solution α_H is an ϵ-approximation to a maximization problem with optimal solution α^* if for all problem instances,

$$\alpha_H \geqslant (1-\epsilon)\alpha^*.]$$

5

For a coalitional game, explain what is meant by the *characteristic function*, an *imputation* and the *core*.

(a) A group of n miners have discovered large and equal sized lumps of gold. Two miners can carry one lump, so that the payoff of a coalition S is

$$v(S) = \begin{cases} |S|/2 & \text{if } |S| \text{ is even} \\ (|S|-1)/2 & \text{if } |S| \text{ is odd}\,. \end{cases}$$

Determine the core in the cases where n is even and where n is odd.

Determine the core if it require three miners to carry one lump.

(b) A pair of shoes consists of a left shoe and a right shoe, and can be sold for £10. Consider a coalitional game with $a+b$ players: a of the players have one left shoe each, and b of the players have one right shoe each. Determine the core for each pair of positive integers (a,b).

6

Let X be a convex set of strategies. Recall that a strategy $x^* \in X$ is an evolutionary stable strategy (ESS) if for every $y \in X$, $y \neq x^*$ then

$$e(x^*, \bar{x}) \;>\; e(y, \bar{x})$$

where $\bar{x} = (1-\epsilon)x^* + \epsilon y$ for sufficiently small $\epsilon > 0$. Briefly discuss the interpretation of $e(.\,,.)$, x^* and y in this definition.

Show that a strategy x^* is an ESS if and only if for every $y \in X$, $y \neq x^*$

$$e(x^*, x^*) \;\geqslant\; e(y, x^*)$$

and if $e(x^*, x^*) = e(y, x^*)$ then

$$e(x^*, y) \;>\; e(y, y)\,.$$

Suppose that a strategy $x \in X$ is a mixture $(p, 1-p)$ of the two pure strategies "Hawk" $= (1,0)$ and "Dove" $= (0,1)$ and that for the pure strategies the pay off matrix is

$$\begin{array}{cc} & \begin{array}{cc} \text{Hawk} & \text{Dove} \end{array} \\ \begin{array}{c} \text{Hawk} \\ \text{Dove} \end{array} & \begin{pmatrix} \frac{1}{2}(V-D) & V \\ 0 & \frac{1}{2}V \end{pmatrix} \end{array}$$

Find an ESS when

(i) $V > D$,

(ii) $V = D$,

(iii) $V < D$,

justifying your answer in each case.

END OF PAPER

UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 6 June, 2011 1:30 pm to 4:30 pm

PAPER 39

MATHEMATICS OF OPERATIONAL RESEARCH

*Attempt no more than **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

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Treasury Tag
Script paper

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None

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(a) Define, for mixed strategies, the *upper value* and *lower value* of a two-person zero-sum game.

(b) Explain how to find the upper and lower values by solving linear programs.

By quoting from the theory of linear programming, explain why they are equal.

(c) Suppose that $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$, A is a $m \times n$ real matrix, and all components of A, b, and c are positive. Consider the two-person zero-sum game in which each player has $m+n+1$ pure strategies and the payoff matrix is

$$M = \left(\begin{array}{c|c|c} 0 & A & -b \\ \hline -A^\top & 0 & c \\ \hline b^\top & -c^\top & 0 \end{array}\right).$$

The first (second) diagonal block is a $m \times m$ ($n \times n$) square matrix of zeros.
What are the upper and lower values of this game?

(d) Suppose that (for both players) an optimal mixed strategy in the zero-sum game with payoff matix M is $\pi^\top = (p_1, \ldots, p_m, q_1, \ldots, q_n, r)$. Prove that $r \neq 0$.

(e) Explain how to find from π an optimal solution to the linear program
LP $= \{\text{maximize}\, c^\top y : Ay \leqslant b,\ y \geqslant 0\}$.

2

(a) Suppose $X \subseteq \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}$, $h : \mathbb{R}^n \to \mathbb{R}^m$ and $b \in \mathbb{R}^m$. Consider the problem

$$\text{P} : \text{minimize } \ f(x) \text{ s.t. } \ h(x) = b, \ x \in X.$$

Formulate the Lagrangian dual problem, P^*.

Show that the optimal value of P^* provides a lower bound on the optimal value of P.

(b) Suppose that A is a $m \times n$ real matrix and $b \in \mathbb{R}^m$. Consider problems:

$$\begin{aligned} \text{QP} : &\underset{x,y}{\text{minimize}} \ \tfrac{1}{2}x^\top x \\ &\text{s.t. } \ x \geqslant 0, \ y \geqslant 0 \\ &Ax - y = b \end{aligned} \qquad\qquad \begin{aligned} \text{QP}^* : &\underset{\lambda,\,\mu,\,x}{\text{maximize}} \ b^\top \lambda - \tfrac{1}{2}x^\top x \\ &\text{s.t. } \ \lambda \geqslant 0, \ \mu \geqslant 0, \\ &x = A^\top \lambda + \mu \end{aligned}$$

Show that QP^* is the Lagrangian dual problem of QP.

(c) Suppose that x, y, λ, μ are feasible for QP and QP^* and such that $\lambda^\top y = 0$ and $x^\top \mu = 0$. Show that these variables provide optimal solutions to QP and QP^*.

(d) Find a matrix M (involving A) and vector q (involving b) such that solutions to QP and QP^* can be found by solving the linear complementarity problem:

$$\text{LCP} : \ \text{Find } w \geqslant 0, \ z \geqslant 0 \text{ s.t. } \ w - Mz = q \text{ and } w^\top z = 0.$$

3

(a) Suppose that A is a $n \times n$ real matrix in which all components are non-negative and $q^\top = (1, \ldots, 1) \in \mathbb{R}^n$. Let

$$S = \{(w, z) : w, z \in \mathbb{R}^n,\ w \geqslant 0,\ z \geqslant 0,\ w + Az = q\}.$$

Explain how Nash equilibria of the two-person bimatrix game in which the payoff matrices for the row and column players are A and $B = A^\top$, respectively, are related to solutions to the linear complementary problem:

$$\textbf{LCP}\text{: Find } (w, z) \in S \text{ such that } w^\top z = 0.$$

(b) Starting from the solution at $(w, z) = (q, 0)$, we wish to find a second solution of **LCP** by using Lemke's algorithm to follow a path through a sequence of points in S, each of which has the property that $i = 1$ is the only index (amongst $\{1, 2, \ldots, n\}$) for which $z_i w_i$ might be non-zero. Arranging your calculations in a tableau, show that with data

$$A = \begin{pmatrix} 3 & 3 & 0 \\ 4 & 0 & 1 \\ 0 & 4 & 5 \end{pmatrix}$$

the path terminates with tableau

w_1	w_2	w_3	z_1	z_2	z_3	
$1/3$	$-1/4$	0	0	1	$-1/4$	$1/12$
0	$1/4$	0	1	0	$1/4$	$1/4$
$-4/3$	1	1	0	0	6	$2/3$

(c) What happens when the choice $i = 1$ is replaced with $i = 3$?

(d) Suppose that for all $(w, z) \in S$ the total number of non-zero components in w and z is at least n. Prove that the number of solutions of **LCP** is even.

(e) Show that there are solutions to this **LCP** that cannot be found by following some Lemke-algorithm path that starts at $(w, z) = (q, 0)$.

4

(a) Given a graph $G = (V, E)$ and a partition of vertices into nonempty sets S and $\bar{S} = V \setminus S$, define the cut value $C(S, \bar{S})$ as the number of edges having one vertex in S and one vertex in $\bar{S}$. In the MIN-CUT decision problem we are given a graph G and integer k and asked if there exists a cut with $C(S, \bar{S}) \leqslant k$. By using what you know about the Ford-Fulkerson algorithm prove that MIN-CUT is in complexity class P.

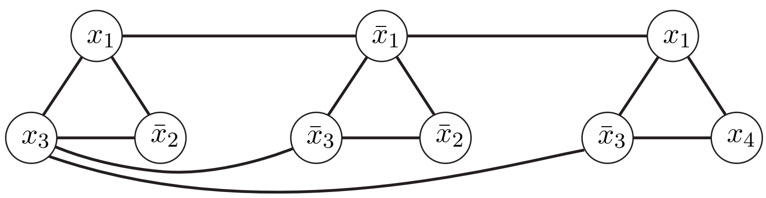
(b) Let MAX-CUT be the problem of finding $\mathrm{OPT}(G) := \max_{S \subset V} C(S, \bar{S})$. Explain how to formulate MAX-CUT as a quadratic programming problem in variables confined to the values 1 and -1.

(c) Consider a Boolean expression, B, that is the conjunction of m clauses, each of which is the disjunction of 3 literals. For example, with $m = 3$ clauses, and $\bar{x}_i =$ 'not x_i',

$$(x_3 \text{ or } x_1 \text{ or } \bar{x}_2) \text{ and } (\bar{x}_3 \text{ or } \bar{x}_1 \text{ or } \bar{x}_2) \text{ and } (\bar{x}_3 \text{ or } x_1 \text{ or } x_4).$$

The NAE-3SAT decision problem asks if it is possible to assign values to the variables (true or false) so that the Boolean expression is true, and also so that the 3 literals in each clause are *not all equal* (i.e. not all true). In the example above, the answer is yes, by taking $x_1 = x_2 = \bar{x}_3 = \bar{x}_4 =$ true.

Let us construct a graph, $H(B)$, in which $3m$ vertices represent the $3m$ literals. Place an edge between any two vertices that represent literals that cannot be equal (such as x_i and $\bar{x}_i$). Suppose this creates K edges. Also place edges between vertices that represent literals in the same clause (giving another $3m$ edges). For the example above, we would have the graph



Use this construction to show that if NAE-3SAT is NP-complete then MAX-CUT is NP-hard. Hint: consider the question: is $\mathrm{OPT}(H(B)) \geqslant K + 2m$?

(d) Consider the following approximation algorithm for MAX-CUT.

Step 1. Arbitrarily partition the vertices into two nonempty sets S and $\bar{S}$.

Step 2. Look for a vertex which if moved from its set to the other set will increase the value of the cut. If no such vertex exists then stop. Otherwise, move this vertex to the other set, and then repeat Step 2.

Let $A(G)$ denote the value of the cut created by this algorithm.
Show that $A(G) \geqslant (1/2)\mathrm{OPT}(G)$. Hint: $\mathrm{OPT}(G) \leqslant |E|$.

5

(a) Let $G = (V, E)$ be a graph with vertex set $V = \{0, \ldots, n\}$. Suppose there is an edge between every pair of vertices, and each edge e has an associated cost $\ell(e)$. Given an edge e, suppose that V can be partitioned into disjoint sets, U and $V \setminus U$, so that e is an edge of least cost between them. Prove that there exists a minimum cost spanning tree that includes e.

(b) Prove that a minimum cost spanning tree can be found by the algorithm which starts with all edges of G coloured white, and at each of n successive steps recolours one edge black, choosing this edge as one of least cost amongst those white edges that could be made black without creating a black cycle.

(c) Suppose that every $\ell(e)$ is a non-negative integer no greater than k. Let S be a nonempty subset of $N = \{1, \ldots, n\}$. As a function of ℓ, let $c(S, \ell)$ denote the least cost of subtree of G which has $|S|$ edges and connects all vertices in S to vertex 0.

For each $j = 1, \ldots k$, define $\ell_j : E \to \{0, 1\}$ by

$$\ell_j(e) = \begin{cases} 0 & \text{if } \ell(e) < j, \\ 1 & \text{if } \ell(e) \geqslant j. \end{cases}$$

Show that $c(S, \ell) = c(S, \ell_1) + \cdots + c(S, \ell_k)$.

(d) In the *minimum cost spanning tree game* the set of players is $N = \{1, \ldots, n\}$ and the characteristic function is defined as $v(S) = c(S, \ell)$, $S \subseteq N$. It is desired to specify a cost sharing, $\{x_{S,i},\ i \in S\}$, for each subset S, having the desirable properties that

$$\sum_{i \in S} x_{S,i} = v(S) \text{ for all } S \subseteq N,\ S \neq \emptyset\,, \tag{1}$$

$$x_{S,i} \geqslant x_{T,i} \text{ for all } i \in S \subset T \subseteq N. \tag{2}$$

Explain why these properties are desirable.

Show that if such numbers exist then $(x_{N,1}, \ldots, x_{N,n})$ is in the core of the game.

(e) Consider a simple case of the above, in which $\ell(e) \in \{0, 1\}$ for all e. Let $S \subseteq N$. For each $i \in S$, set $x_{S,i} = 0$ if, for some $j \in S \cup \{0\}$ and $j < i$, vertex i can be connected to j by a path of cost 0 passing through only vertices in $S \cup \{0\}$. Otherwise set $x_{S,i} = 1$. Prove that with this definition (1) and (2) hold.

How could you solve (1)–(2) in a case that $\ell(e) \in \{0, \ldots, k\}$ for all $e \in E$?

6

(a) A item is to be auctioned between two identical risk-neutral bidders, who have private valuations for winning the item that are independently distributed uniformly on $[0,1]$. The auction design specifies that the item is won by the highest bidder, and, as functions of their bids, the bidders shall make certain non-negative payments to the auctioneer. All the above (apart from the private valuations) is public knowledge.

Suppose that in equilibrium the optimal strategy of bidder i, when having private valuation v_i, is to participate in the auction and bid $b(v_i)$ if $v_i \geqslant \bar{v}_i$, but to not participate if $v_i < \bar{v}_i$, where $\bar{v}_i$ is some number in $[0,1]$. Conditional on v_i, let $\pi(v_i)$ and $e(v_i)$ denote, respectively, bidder i's expected profit and expected payment.

Explain why $\pi(v_i) = v_i^2 - e(v_i)$ for $v_i \geqslant \bar{v}_i$, and $\pi(0) = 0$ for $v_i < \bar{v}_i$. Show that

$$\pi(v_i) = (1/2)(v_i^2 - \bar{v}_i^2), \quad v_i \geqslant \bar{v}_i.$$

(b) Consider three auctions designs in which: (i) the winner pays his bid, (ii) the loser pays his bid, (iii) both winner and loser pay their bids. Why in these designs is $\bar{v}_i = 0$?

Find $b(v_i)$ in each case. Verify that in (ii) $b(v_i) \to \infty$ as $v_i \to 1$.

Verify that (i) and (iii) guarantee the same expected revenue for the auctioneer.

(c) Show that if we add to auction (iii) the rule that the minimum permitted bid is $1/4$ then $\bar{v}_1 = \bar{v}_2 = 1/2$.

Show that the expected revenue obtained by the auctioneer in this auction exceeds that in any of (i), (ii) and (iii).

END OF PAPER

UNIVERSITY OF CAMBRIDGE

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PAPER 39

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$\mathrm{LP} = \{\text{maximize}\, c^\top y : Ay \leqslant b,\ y \geqslant 0\}$.

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$$\mathrm{P} : \text{minimize } \; f(x) \text{ s.t. } h(x) = b, \; x \in X.$$

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Show that the optimal value of P^* provides a lower bound on the optimal value of P.

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Show that QP^* is the Lagrangian dual problem of QP.

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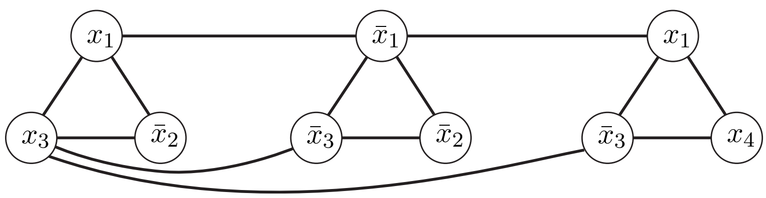

Use this construction to show that if NAE-3SAT is NP-complete then MAX-CUT is NP-hard. Hint: consider the question: is $\mathrm{OPT}(H(B)) \geqslant K + 2m$?

(d) Consider the following approximation algorithm for MAX-CUT.

Step 1. Arbitrarily partition the vertices into two nonempty sets S and $\bar{S}$.

Step 2. Look for a vertex which if moved from its set to the other set will increase the value of the cut. If no such vertex exists then stop. Otherwise, move this vertex to the other set, and then repeat Step 2.

Let $A(G)$ denote the value of the cut created by this algorithm.
Show that $A(G) \geqslant (1/2)\mathrm{OPT}(G)$. Hint: $\mathrm{OPT}(G) \leqslant |E|$.

5

(a) Let $G = (V, E)$ be a graph with vertex set $V = \{0, \ldots, n\}$. Suppose there is an edge between every pair of vertices, and each edge e has an associated cost $\ell(e)$. Given an edge e, suppose that V can be partitioned into disjoint sets, U and $V \setminus U$, so that e is an edge of least cost between them. Prove that there exists a minimum cost spanning tree that includes e.

(b) Prove that a minimum cost spanning tree can be found by the algorithm which starts with all edges of G coloured white, and at each of n successive steps recolours one edge black, choosing this edge as one of least cost amongst those white edges that could be made black without creating a black cycle.

(c) Suppose that every $\ell(e)$ is a non-negative integer no greater than k. Let S be a nonempty subset of $N = \{1, \ldots, n\}$. As a function of ℓ, let $c(S, \ell)$ denote the least cost of subtree of G which has $|S|$ edges and connects all vertices in S to vertex 0.

For each $j = 1, \ldots k$, define $\ell_j : E \to \{0, 1\}$ by

$$\ell_j(e) = \begin{cases} 0 & \text{if } \ell(e) < j, \\ 1 & \text{if } \ell(e) \geqslant j. \end{cases}$$

Show that $c(S, \ell) = c(S, \ell_1) + \cdots + c(S, \ell_k)$.

(d) In the *minimum cost spanning tree game* the set of players is $N = \{1, \ldots, n\}$ and the characteristic function is defined as $v(S) = c(S, \ell)$, $S \subseteq N$. It is desired to specify a cost sharing, $\{x_{S,i},\ i \in S\}$, for each subset S, having the desirable properties that

$$\sum_{i \in S} x_{S,i} = v(S) \text{ for all } S \subseteq N,\ S \neq \emptyset\,, \tag{1}$$

$$x_{S,i} \geqslant x_{T,i} \text{ for all } i \in S \subset T \subseteq N. \tag{2}$$

Explain why these properties are desirable.

Show that if such numbers exist then $(x_{N,1}, \ldots, x_{N,n})$ is in the core of the game.

(e) Consider a simple case of the above, in which $\ell(e) \in \{0, 1\}$ for all e. Let $S \subseteq N$. For each $i \in S$, set $x_{S,i} = 0$ if, for some $j \in S \cup \{0\}$ and $j < i$, vertex i can be connected to j by a path of cost 0 passing through only vertices in $S \cup \{0\}$. Otherwise set $x_{S,i} = 1$. Prove that with this definition (1) and (2) hold.

How could you solve (1)–(2) in a case that $\ell(e) \in \{0, \ldots, k\}$ for all $e \in E$?

6

(a) A item is to be auctioned between two identical risk-neutral bidders, who have private valuations for winning the item that are independently distributed uniformly on $[0,1]$. The auction design specifies that the item is won by the highest bidder, and, as functions of their bids, the bidders shall make certain non-negative payments to the auctioneer. All the above (apart from the private valuations) is public knowledge.

Suppose that in equilibrium the optimal strategy of bidder i, when having private valuation v_i, is to participate in the auction and bid $b(v_i)$ if $v_i \geqslant \bar{v}_i$, but to not participate if $v_i < \bar{v}_i$, where $\bar{v}_i$ is some number in $[0,1]$. Conditional on v_i, let $\pi(v_i)$ and $e(v_i)$ denote, respectively, bidder i's expected profit and expected payment.

Explain why $\pi(v_i) = v_i^2 - e(v_i)$ for $v_i \geqslant \bar{v}_i$, and $\pi(0) = 0$ for $v_i < \bar{v}_i$. Show that

$$\pi(v_i) = (1/2)(v_i^2 - \bar{v}_i^2), \quad v_i \geqslant \bar{v}_i.$$

(b) Consider three auctions designs in which: (i) the winner pays his bid, (ii) the loser pays his bid, (iii) both winner and loser pay their bids. Why in these designs is $\bar{v}_i = 0$?

Find $b(v_i)$ in each case. Verify that in (ii) $b(v_i) \to \infty$ as $v_i \to 1$.

Verify that (i) and (iii) guarantee the same expected revenue for the auctioneer.

(c) Show that if we add to auction (iii) the rule that the minimum permitted bid is $1/4$ then $\bar{v}_1 = \bar{v}_2 = 1/2$.

Show that the expected revenue obtained by the auctioneer in this auction exceeds that in any of (i), (ii) and (iii).

END OF PAPER