MARKOV CHAINS — Richard Weber

Example Sheet 2

1. The rooted binary tree is an infinite graph T with one distinguished vertex R from which comes a single edge; at every other vertex there are three edges and there are no closed loops. The random walk on T jumps from a vertex along each available edge with equal probability. Show that the random walk is transient.

2. Show that the simple symmetric random walk in $\mathbb{Z}^4$ is transient.

3. Show that for the Markov chain $(X_n)_{n\geq 0}$ in Example 12 from Sheet 1

$$P(X_n \to \infty \text{ as } n \to \infty) = 1\,.$$

Suppose the transition probabilities satisfy instead

$$p_{ii+1} = \left(\frac{i+1}{i}\right)^{\alpha} p_{ii-1}$$

for some $\alpha \in (0,\infty)$. What then is the value of $P(X_n \to \infty \text{ as } n \to \infty)$?

4. Find all invariant distributions of the transition matrix

$$P = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

5. Gas molecules move about randomly in a box which is divided into two halves symmetrically by a partition. A hole is made in the partition. Suppose there are N molecules in the box. Show that the number of molecules on one side of the partition just after a molecule has passed through the hole evolves as a Markov chain. What are the transition probabilities? What is the invariant distribution of this chain?

6. A particle moves on the eight vertices of a cube in the following way: at each step the particle is equally likely to move to each of the three adjacent vertices, independently of its past motion. Let i be the initial vertex occupied by the particle, o the vertex opposite i. Calculate each of the following quantities:

(i) the expected number of steps until the particle returns to i;

(ii) the expected number of visits to o until the first return to i;

(iii) the expected number of steps until the first visit to o.

7. Find the invariant distributions of the transition matrices in Example 7 from Sheet 1, parts (a), (b) and (c), and use them to check your answers.

8. A fair die is thrown repeatedly. Let X_n denote the sum of the first n throws. Find

$$\lim_{n\to\infty} \mathbb{P}(X_n \text{ is a multiple of } 13).$$

9. Each morning a student takes one of the three books from her shelf. The probability that she chooses book i is $\alpha_i, 0 < \alpha_i < 1, i = 1, 2, 3$, and choices on successive days are independent. In the evening she replaces the book at the left-hand end of the shelf. If p_n denotes the probability that on day n the student finds the books in the order 1,2,3, from left to right, show that, irrespective of the initial arrangement of the books, p_n converges to a limit, say π_{123}, as $n \to \infty$, and determine π_{123}.

Suppose there are n books. We say '*i precedes j*' if book i lies to the left of book j. Why, in equilibrium, does $P(i \text{ precedes } j) = \alpha_i/(\alpha_i + \alpha_j)$? Write down an expression for $\pi_{123\ldots n}$.

10. In each of the following cases determine whether the stochastic matrix P is reversible:

(a) $\begin{pmatrix} p & 1-p \\ q & 1-q \end{pmatrix}$ (b) $\begin{pmatrix} 0 & p & 1-p \\ 1-p & 0 & p \\ p & 1-p & 0 \end{pmatrix}$

(c) $I = \{0, 1, \ldots, N\}$ and $p_{ij} = 0$ if $|j - i| \geq 2$;

(d) $I = \{0, 1, 2, \ldots\}$ and $p_{01} = 1$, $p_{ii+1} = p$, $p_{ii-1} = 1 - p$ for $i \geq 1$.

11. A professor has N umbrellas, which he keeps either at home or in his office. He walks to and from his office each day, and takes an umbrella with him if and only if it is raining. Throughout each journey, it either rains, with probability p, or remains fine, with probability $1 - p$, independently of the past weather. What is the long run proportion of journeys on which he gets wet?

12. (Renewal theorem) Let $Y_1, Y_2, \ldots$ be independent, identically distributed random variables with values in $\{1, 2, \ldots\}$. Suppose that the set of integers

$$\{n : P(Y_1 = n) > 0\}$$

has greatest common divisor 1. Set $\mu = \mathbb{E}(Y_1)$. Show that the following process is a Markov chain:

$$X_n = \inf\{m \geq n : m = Y_1 + \cdots + Y_k \text{ for some } k \geq 0\} - n.$$

Determine

$$\lim_{n \to \infty} P(X_n = 0)$$

and hence show that as $n \to \infty$

$$P(n = Y_1 + \cdots + Y_k \text{ for some } k \geq 0) \to 1/\mu.$$

13. An opera singer is due to perform a long series of concerts. Having a fine artistic temperament, she is liable to pull out each night with probability 1/2. Once this has happened she will not sing again until the promoter convinces her of his high regard. This he does by sending flowers every day until she returns. Flowers costing x thousand pounds, $0 \leq x \leq 1$, bring about a reconciliation with probability $\sqrt{x}$. The promoter stands to make £750 from each successful concert. How much should he spend on flowers?

Extras

The following questions are optional, but could be fun to do or to discuss with your supervisor. You should at least read them, to appreciate the type of problems that the methods of our course can address.

14. We wish to simulate a random variable X for which we know $P(X = i) \propto \alpha_i$, $i \in I$, but we do not know the constant of proportionality because $\sum_{i \in I} \alpha_i$ is too difficult calculate. Let $P = (p_{ij})$ be the (known) transition matrix of some irreducible Markov chain on I. Define

$$q_{ij} = \min\{p_{ij}, \alpha_j p_{ji}/\alpha_i\}, \ i \neq j, \quad \text{and} \quad q_{ii} = 1 - \sum_{j \neq i} q_{ij}.$$

Prove that $Q = (q_{ij})$ is the transition matrix of a reversible Markov chain with invariant distribution proportional to α. How can this help with the problem of simulating X?

15. Let $P = (p_{ij})$ be the transition matrix of an irreducible finite-state Markov chain, with all entries rational numbers. A class of n children plays a game of successive rounds. Each child starts with some number of gumdrops.

Suppose that at the end of a round child i has m_i gumdrops. The teacher then hands out gumdrops so that each child has n_i gumdrops, where n_i is the least number not below m_i such that $p_{ij}n_i$ is an integer, for every j. Finally, for each i and j, child i passes $p_{ij}n_i$ gumdrops to child j.

Carry out this algorithm when $n = 3$,

$$P = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & 0 & \frac{1}{3} \end{pmatrix}$$

and each of the 3 children starts with 3 gumdrops. You should find that the algorithm terminates after 3 rounds.

What do the final numbers of gumdrops tell you about the Markov chain?

Prove that the algorithm always terminates in a finite number of rounds.

16. In Question 9 the three books were being replaced on the shelf using a *move-to-front rule*. Now consider a different rule: whenever a book is retrieved from position i (≥ 2) (from the left hand end of the shelf) then it is returned to position $i-1$. If the book at the left hand end ($i = 1$) is selected then it is returned there. This is called the *transposition rule*. Show that the resulting Markov chain is reversible, with an invariant distribution $\{\pi_{123}, \pi_{132}, \pi_{213}, \pi_{231}, \pi_{312}, \pi_{321}\}$ such that $\pi_{ijk} \propto \alpha_i^2\alpha_j$.

The 'average retrieval cost' (searching from the left hand end) is defined as

$$\sum_{\substack{i,j,k\in\{1,2,3\} \\ i\neq j\neq k}} \pi_{ijk}(\alpha_i + 2\alpha_j + 3\alpha_k) \quad \Big(= 1 + \sum_{i\neq j} \alpha_i P(j \text{ precedes } i)\Big).$$

Show that for all $\alpha_1, \alpha_2, \alpha_3$ the transposition rule has smaller average retrieval cost than the move-to-front rule.

[*Hint. Show that for the transposition rule:* $P(i \text{ precedes } j)/P(j \text{ precedes } i) > \alpha_i/\alpha_j$ *when* $\alpha_i > \alpha_j$. *Use this to show that* $\alpha_j P(i \text{ precedes } j) + \alpha_i P(j \text{ precedes } i) \leq 2\alpha_i\alpha_j/(\alpha_i + \alpha_j)$ *for all* $i \neq j$.]

17. In the classic coupon-collector problem a collector seeks to obtain a complete set of n coupons, or prizes. Each box of cereal that the collector buys is equally likely to contain one of the n prizes. Let's consider a slightly more complicated model. Suppose $n = 3$, and the prizes are A, B and C and cereal boxes are of 3 types, distinguishable by their outer packaging. In box type (AB) the prize is equally likely to be A or B. In box type (BC) the prize is equally likely to be B or C. Similarly, in box type (AC) the prize is equally likely to be A or C. The aim is to collect N complete sets and minimize the leftovers. Clearly, we need to buy at least $3N$ boxes.

Suppose the collector has reached a point at which she has m complete sets, with i A, j B and 0 C as leftovers, $i \geq j$. Obviously she should buy box type (BC). Similarly, she always buys the box type that has the two prizes that are least represented amongst her leftovers. Represent the state of the leftovers as a Markov chain, prove it is recurrent and that as $N \to \infty$ the expected number of leftovers at the point she completes the Nth complete set tends to the golden ratio $\frac{1}{2}(1 + \sqrt{5})$.

[*Hint: In representing the process by a Markov chain you should see that it is enough to take the state to be the current number of leftovers. For if this number is even then the state must be of the form* $(i, i, 0)$, *and if odd of the form* $(i+1, i, 0)$, $i = 0, 1, \ldots$.]